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Brief paper

Stability analysis and stabilization control of multi-variable switched stochastic systems☆

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Abstract

In this paper, the mean square (MS) stability and exponential mean square (EMS) stability of multi-variable switched stochastic systems are investigated. Based on the concept of the average dwell-time and the ratio of the total time running on all unstable subsystems to the total time running on all stable subsystems, some sufficient conditions are given to ensure the MS stability and EMS stability of the switched stochastic systems involved. Further, for the switched stochastic control systems with all subsystems controllable or stabilizable, EMS stabilization controls and sufficient conditions on EMS stabilization are presented, and the convergent rates of the closed-loop systems are obtained.

Keywords: Switched system; Stochastic system; Average dwell-time; Mean square stability; Stabilization control

1. Introduction

Since switched control systems exist widely in engineering technology and social systems (Mariton, 1990), the analysis and control problems of them have attracted extensive attention from many researchers. Many important progress and remarkable achievements have been made on issues such as controllability, reachability and stabilizability (Sun, Ge, & Lee, 2002; Xie & Wang, 2003), control and switching law design (Cheng, Guo, Lin, & Wang, 2005; Daafouz, Riedinger, & Iung, 2002; Ishii & Francis, 2002; Liberzon & Morse, 1999; Li, Wen, & Soh, 2001), optimal control (Giua, Seatzu, & Van Der Mee, 2001) and so on. Among others, stability analysis and stabilization control are two hot topics. The basic problems considered include stability analysis for systems with specific switching laws (Branicky, 1998), stability analysis for systems with arbitrary switching laws (Liberzon, Hespanha, & Morse, 1999), and design of stabilization switching laws (Daafouz et al., 2002; Li et al., 2001), etc. For the first topic, the concepts such as dwell-time and average dwell-time, etc. are introduced by Hespanha and Morse (1999), and then, used in Zhai, Hu, Yasuda, and Michel (2001) and Hespanha (2004), which make it possible for us to analyze the stability properties of switched systems with stable and unstable subsystems. For the second topic, common Lyapunov function and linear matrix inequality were used for stabilization control of linear switched systems with arbitrary switching laws (Liberzon et al., 1999). And the concepts of switching frequency and dwell-time were used for stabilization control of switched systems with controllable and uncontrollable subsystems in Cheng et al. (2005), where a subtle capacity characterization of feedback matrix on improving the convergent rate of the closed-loop linear time-invariant systems is provided, which is the key to our success of stabilization control for switched stochastic (SS) systems (see Section 4). The works mentioned above mainly focused on deterministic systems. For stochastic switched control systems, although some results have been given (Caines & Zhang, 1996, and the references therein), they are merely concentrated on the case where the switching laws are random, Markovian processes with known transition probability.

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The systems studied in this paper are with deterministic switching laws and unknown stochastic disturbances. By describing quantitatively the quadratic stability and instability of matrices, based on the concept of the average dwell-time and the ratio of the total time running on all unstable subsystems to the total time running on all stable subsystems, some sufficient conditions on mean square (MS) stability and exponential mean square (EMS) stability are given, respectively. In addition, by analyzing the impact of the uncontrollable part of an uncontrollable but stabilizable (UCbS) stochastic system on the convergent rate of the closed-loop system, the stabilization control problems are investigated for two classes of SS control systems, one is with all subsystems controllable, and the other is with some subsystems controllable and some UCbS. Some sufficient conditions ensuring the EMS stability of the closed-loop systems are obtained, and the convergent rates of the closed-loop systems are analyzed. Unlike conventional time-varying parameter systems, here it is only required that the switching law $\sigma(t)$ is observable. In other words, for control designers, at any time instance $t > t_0$, only $\sigma(s)$, $\forall s \in [t_0, t]$ is known and available for control design. As for $\sigma(s)$ for s in (t, ∞), the designers probably do not know any information before the time instance t.

The remainder of this paper is organized as follows. In Section 2, some notations and definitions are introduced and the problems to be studied are formulated. In Section 3, the stability of autonomous (i.e. control-free) SS systems is analyzed, and some sufficient conditions on MS stability and EMS stability are given, respectively. In Section 4, the stabilization problem of SS systems is investigated, some sufficient conditions on EMS stability and the convergent rate of the closed-loop systems are obtained. Section 5 concludes the paper.

2. Notations and problem formulation

Consider the following SS system

$$\begin{aligned} \mathrm{d}x(t) = {} & A_{\sigma(t)}x(t)\,\mathrm{d}t + B_{\sigma(t)}u_{\sigma(t)}(t)\,\mathrm{d}t \\ & + \alpha(t, x(t))\,\mathrm{d}W(t) \end{aligned} \tag{1}$$

with initial condition $x(t_0)=x_0$, where function $\sigma(\cdot)$: $[t_0, \infty) \to \mathscr{I} = \{1, 2, \dots, N\}$ is the switching law and is deterministic (namely, not random), piecewise constant and right continuous; $A_i \in \mathbb{R}^{n\times n}$ and $B_i \in \mathbb{R}^{n\times m_i}$ for $i \in \mathscr{I}$ are constant matrices; $x(t) \in \mathbb{R}^n$, $u_i \in \mathbb{R}^{m_i}$ $(i \in \mathscr{I})$ are system state and inputs, respectively; $\alpha(t, x) = (\alpha_1(t, x), \dots, \alpha_k(t, x))^{\mathrm{T}} \in \mathbb{R}^{n\times k}$ is the noise coefficient, continuous on (t, x) and uniformly Lipschitz continuous on x; $W(t) \in \mathbb{R}^k$ is a standard Brownian motion.

Throughout the paper, $\mathbb{R}^{n\times m}$ denotes the set of all $n \times m$ real matrices; $E(\cdot)$ denotes the mathematical expectation; $\|\cdot\|$ denotes the Euclidean norm in the n-dimensional real space $\mathbb{R}^n$ and the corresponding induced matrix norm; $\Lambda(A)$ denotes the set of all eigenvalues of a square matrix A, and for a real symmetric matrix A, $\lambda_M(A)$ and $\lambda_m(A)$ denotes its maximal eigenvalue and minimal eigenvalue, respectively; $\mathbb{C}^-$ denotes the open left half complex plane; I denotes the identity matrix with proper dimension; for any given matrix or vector X, X^{T} denotes its transpose.

This paper investigates the MS stabilization control of the system (1). To do so, we introduce some concepts on MS stability for the following stochastic systems:

$$\mathrm{d}x(t) = A_{\sigma(t)}x(t)\,\mathrm{d}t + \alpha(t, x(t))\,\mathrm{d}W(t), \quad x(t_0) = x_0. \tag{2}$$

Definition 1. For System (2), if its solution process $\{x(t), t \geqslant t_0\}$ with $x(t_0)=x_0$ is existent and unique, and satisfies $\sup_{t\geqslant t_0} E\,\|x(t)\|^2 < \infty$, then the solution process is said MS stable; in addition, if for some positive constants M_0 and a, $E\|x(t)\|^2 \leqslant M_0 \mathrm{e}^{-a(t-t_0)}$, $\forall\, t \geqslant t_0$, then the solution process is said EMS stable (with convergent rate a).

System (2) is said MS stable if for $\forall x_0 \in \mathbb{R}^n$, the $\{x(t), t \geqslant t_0\}$ with $x(t_0) = x_0$ is MS stable, and is said EMS stable if for $\forall x_0 \in \mathbb{R}^n$, the $\{x(t), t \geqslant t_0\}$ with $x(t_0) = x_0$ is EMS stable.

Definition 2. The problem to design a control law $u_{\sigma(t)}(t)$ such that System (1) is MS (or EMS) stable is said MS (correspondingly, EMS) stabilization control problem.

Definition 3 (*Hespanha & Morse, 1999*). For any given $\sigma(t)$ and $t > s \geqslant t_0$, let $N_\sigma(s, t)$ be the switching number of $\sigma(t)$ on interval $[s, t)$. For any given $N_0 > 0$ and $\tau^* > 0$, set $\mathscr{S}[\tau^*, N_0] = \{\sigma(\cdot) : N_\sigma(s, t) \leqslant N_0 + (t - s)/\tau^*, \forall t \geqslant t_0, \forall s \in [t_0, t)\}$. Then τ^* and N_0 are called average dwell-time and chatter bound of $\mathscr{S}[\tau^*, N_0]$, respectively. For a given $\sigma(t) \in \mathscr{S}[\tau^*, N_0]$, the constant τ_σ determined by $1/\tau_\sigma = \sup_{s\geqslant t_0} \sup_{t>s} (N_\sigma(s, t) - N_0)/(t - s)$ is called the average dwell-time of $\sigma(t)$.

Remark 1. In fact, for any given $N_0 > 0$, the average dwell-time $\tau^* > 0$ in Definition 3 is the infimum of the average dwell-time τ_σ of all $\sigma(t) \in \mathscr{S}[\tau^*, N_0]$ and we have

$$N_\sigma(s, t) \leqslant N_0 + (t - s)/\tau_\sigma. \tag{3}$$

3. Mean square stability of SS systems

For studying the MS and EMS stability of System (2), we first characterize the quadratically stable convergence rate of stable matrices.

For any given $n \times n$ positive definite (PD) matrix P_0, if $A_i^{\mathrm{T}} P_0 + P_0 A_i < 0$, then along the trajectory of $\dot{x} = A_i x$ Lyapunov function $V(x) = x^{\mathrm{T}} P_0 x$ satisfies $\dot{V} = x^{\mathrm{T}}[A_i^{\mathrm{T}} P_0 + P_0 A_i]x \leqslant \gamma V(x)$, where $\gamma = -[\lambda_M(A_i^{\mathrm{T}} P_0 + P_0 A_i)]/\lambda_M(P_0) > 0$. This implies that $V(x) \leqslant V(x(t_0))\mathrm{e}^{-\gamma(t-t_0)}$, $\forall t \geqslant t_0$. Here we name this γ as the convergent rate of quadratic stability of A_i.

Inspired by this, we determine the largest γ with respect to all PD matrices P. For any given $\alpha \in (0, 1)$, set $\mathscr{D} = \{P \in \mathbb{R}^{n\times n} : P \geqslant 0,\ \|P\| = 1\}$, $\mathscr{D}_\alpha = \{P \in \mathbb{R}^{n\times n} :\ P \geqslant \alpha I,\ \|P\| = 1\}$. For any given $n \times n$ matrix set $\mathscr{A} = \{A_i, i \in \mathscr{I}\}$, let

$$a_i(\mathscr{A}) = \begin{cases} \min\limits_{P\in\mathscr{D}} \lambda_M(PA_i + A_i^{\mathrm{T}} P), & \Lambda(A_i) \subset \mathbb{C}^-, \\ \max\limits_{P\in\mathscr{D}_\alpha} \lambda_M(PA_i + A_i^{\mathrm{T}} P), & \Lambda(A_i) \not\subset \mathbb{C}^-; \end{cases} \tag{4}$$

for any given stable matrix $A_i \in \mathscr{A}$, let $\mathscr{P}_{i1}(\mathscr{A})=\{P\in\mathscr{D}:\lambda_M(PA_i+A_i^{\mathrm{T}}P)=a_i(\mathscr{A})\}$, for any given unstable matrix $A_i\in\mathscr{A}$, let $\mathscr{P}_{i2}(\mathscr{A})=\{P\in\mathscr{D}_\alpha:\lambda_M(PA_i+A_i^{\mathrm{T}}P)=a_i(\mathscr{A})\}$. Denote

$$\mathscr{P}_i(\mathscr{A})=\begin{cases}\mathscr{P}_{i1}(\mathscr{A}), & \Lambda(A_i)\subset\mathbb{C}^-,\\ \mathscr{P}_{i2}(\mathscr{A}), & \Lambda(A_i)\not\subset\mathbb{C}^-;\end{cases} \tag{5}$$

$$\begin{cases}a^+(\mathscr{A})=\max\limits_{i\in\mathscr{I}}\max\,(a_i(\mathscr{A}),0),\\ a^-(\mathscr{A})=\min\limits_{i\in\mathscr{I}}\max\,(|a_i(\mathscr{A})|,0);\end{cases} \tag{6}$$

$$\lambda_m=\min\left(\min_{i\in\mathscr{I}}\inf_{P\in\mathscr{P}_i(\mathscr{A})}\lambda_m(P),\ \min_{P\in\mathscr{D}_\alpha}\lambda_m(P)\right), \tag{7}$$

$$\lambda_M=\max_{i\in\mathscr{I}}\sup_{P_i\in\mathscr{P}_i(\mathscr{A})}\lambda_M(P),\quad h=\ln\frac{\lambda_M}{\lambda_m}. \tag{8}$$

In the following, we will denote $a_i(\mathscr{A})$, $\mathscr{P}_i(\mathscr{A})$, $a^+(\mathscr{A})$ and $a^-(\mathscr{A})$ by a_i, $\mathscr{P}_i$, a^+ and a^- for short, respectively.

Remark 2. All sets $\mathscr{P}_i$ $(i\in\mathscr{I})$ are nonempty, since $\mathscr{D}$ and $\mathscr{D}_\alpha$ are bounded and closed, and $\lambda_M(PA_i+A_i^{\mathrm{T}}P)$ is continuous and homogeneous on P. Besides, it is easy to show that for $\forall P_i\in\mathscr{P}_i$, $\lambda_M=1$, $P_iA_i+A_i^{\mathrm{T}}P_i\leqslant a_iI$.

Remark 3. When $\Lambda(A_i)\subset\mathbb{C}^-$, the a_i defined on $\mathscr{D}$ in (4) can uniquely measure the maximal convergence rate of quadratic stability of matrix A_i in some sense. Actually, noticing that $P_0/\lambda_M(P_0)\in\mathscr{D}$, and by (4) we have

$$\gamma=-\lambda_M\left(A_i^{\mathrm{T}}\frac{P_0}{\lambda_M(P_0)}+\frac{P_0}{\lambda_M(P_0)}A_i\right)\leqslant -a_i.$$

This shows that, for any PD matrix P_0, if A_i is quadratic stable, then the convergent rate is not greater than $-a_i$.

Definition 4. For System (2), given matrix set $\mathscr{A}=\{A_i,i\in\mathscr{I}\}$ and switching law $\sigma(t)$, let $T_\sigma^-(s,t;\mathscr{A})$ and $T_\sigma^+(s,t;\mathscr{A})$ denote, respectively, the total time running on all stable subsystems and the total time running on all unstable subsystems of System (2) in $[s,t)$; and for $\forall a^*\in(0,a^-]$ and $\tau^*>0$, define

$$\mathscr{S}_1[a^*,\tau^*;\mathscr{A}] =\left\{\sigma(\cdot):\sup_{t>s\geqslant t_0}\frac{T_\sigma^+(s,t;\mathscr{A})}{T_\sigma^-(s,t;\mathscr{A})}\leqslant\frac{a^--a^*}{a^++a^*},\tau_\sigma\geqslant\tau^*\right\},$$

$$\mathscr{S}_2[a^*,\tau^*;\mathscr{A}]=\{\sigma(\cdot):\ \sigma(t)\in\mathscr{S}_1[a^*,\tau^*;\mathscr{A}],\tau_\sigma>\tau^*\}.$$

Remark 4. By Definition 4, for $\forall\sigma(t)$ in $\mathscr{S}_1[a^*,\tau^*;\mathscr{A}]$ (or $\mathscr{S}_2[a^*,\tau^*;\mathscr{A}]$), the ratio of $T_\sigma^+(s,t;\mathscr{A})$ to $T_\sigma^-(s,t;\mathscr{A})$ has an upper bound $(a^--a^*)/(a^++a^*)$, or equivalently,

$$a^+T_\sigma^+(s,t;\mathscr{A})-a^-T_\sigma^-(s,t;\mathscr{A}) \leqslant -a^*(t-s)\quad\forall t>s\geqslant t_0.$$

In particular, when all A_i $(i\in\mathscr{I})$ are stable, $\mathscr{S}_1[a^*,\tau^*;\mathscr{A}]$ (and $\mathscr{S}_2[a^*,\tau^*;\mathscr{A}]$) includes all $\sigma(t)$ with average dwell-time no less than (correspondingly, larger than) τ^*.

Lemma 1. *Consider the following stochastic system*:

$$\mathrm{d}x(t)=Ax(t)\,\mathrm{d}t+\alpha(t,x(t))\,\mathrm{d}W(t),\quad x(t_0)=x_0,$$

where $A\in\mathbb{R}^{n\times n}$ is a constant matrix, $W(t)$ and $\alpha(t,x)$ are similar to (1). *If the initial value x_0 satisfies $E\|x_0\|^2<\infty$ and $\alpha_r(t,x)$ $(r=1,2,\ldots,k)$ satisfies*

$$\|\alpha_r(t,x(t))\|\leqslant c\|x(t)\|+b(t)\quad\forall t\geqslant t_0, \tag{9}$$

where c is a nonnegative constant, $b(t)$ is random process making $Eb^2(t)$ locally Lebesgue integrable on $[t_0,\infty)$, then the $\{x(t),t\geqslant t_0\}$ of this system is existent and unique, and for any PD matrix $P\in\mathbb{R}^{n\times n}$,

$$E[x^{\mathrm{T}}(t)Px(t)]\leqslant \mathrm{e}^{a(t-t_0)}E[x_0^{\mathrm{T}}Px_0] +2k\lambda_M(P)\int_{t_0}^{t}\mathrm{e}^{a(t-s)}Eb^2(s)\,\mathrm{d}s, \tag{10}$$

where $a=[\lambda_M(PA+A^{\mathrm{T}}P)+2k\lambda_M(P)c^2]/\lambda_m(P)$.

Proof. From Friedman (1975) it can be seen that $\{x(t),t\geqslant t_0\}$ is existent and unique. Then, notice the independent increment property of Brownian motion, by Itô formula and Bellman–Gronwall inequality, one can get (10). □

Corollary 1. *Under the conditions of Lemma* 1 *with $b(t)\equiv 0$ in* (9), *then $E[x^{\mathrm{T}}(t)Px(t)]\leqslant E[x_0^{\mathrm{T}}Px_0]\mathrm{e}^{a(t-t_0)}$, where $a=[\lambda_M(PA+A^{\mathrm{T}}P)+k\lambda_M(P)c^2]/\lambda_m(P)$.*

Lemma 2. *For System* (2), *if $E\|x_0\|^2<\infty$ and $\alpha(t,x)$ satisfies* (9), *then for $\forall\sigma(t)\in\mathscr{S}_1[a^*,\tau^*;\mathscr{A}]$, the $\{x(t),\ t\geqslant t_0\}$ of System* (2) *is existent*, *unique and has*

$$E\|x(t)\|^2\leqslant \mathrm{e}^{h+\Pi(t_0,t)}E\|x_0\|^2 +2k\int_{t_0}^{t}\mathrm{e}^{h+\Pi(s,t)}Eb^2(s)\,\mathrm{d}s, \tag{11}$$

where $\Pi(s,t)=hN_\sigma(s,t)+(2kc^2-a^*)(t-s)/\lambda_m$, the constants a^-, λ_m, h and c are given by (6)–(9).

Proof. The existence and uniqueness of solution process $\{x(t),t\geqslant t_0\}$ follows from Friedman (1975), so we need only to prove (11). For any given $t>t_0$, suppose that $\sigma(t)$ has j switching points in $[t_0,t)$, and denote these j switching points by $t_1,\ldots,t_j$, respectively; and for $l=0,1,\ldots,j$, let $\mu_{t_l}=[\lambda_M(A_{\sigma(t_l)}^{\mathrm{T}}P_{\sigma(t_l)}+P_{\sigma(t_l)}A_{\sigma(t_l)})+2kc^2]/\lambda_m(P_{\sigma(t_l)})$. Noticing that $\lambda_M(P_{\sigma(t_l)})=1$, by Lemma 1 we have $E\{x^{\mathrm{T}}(t)P_{\sigma(t_j)}x(t)\}\leqslant \mathrm{e}^{\mu_{t_j}(t-t_j)}E\|x(t_j)\|^2+2k\int_{t_j}^{t}\mathrm{e}^{\mu_{t_j}(t-s)}Eb^2(s)\,\mathrm{d}s$, which together with (7) gives

$$E\|x(t)\|^2\leqslant\lambda_m^{-1}\left(\mathrm{e}^{\mu_{t_j}(t-t_j)}E\|x(t_j)\|^2 +2k\int_{t_j}^{t}\mathrm{e}^{\mu_{t_j}(t-s)}Eb^2(s)\,\mathrm{d}s\right). \tag{12}$$

Similarly, for $l=1,2,\ldots,j$ we can obtain

$$E\|x(t_l)\|^2 \leqslant \lambda_m^{-1}\left(\mathrm{e}^{\mu_{t_{l-1}}(t_l-t_{l-1})}E\|x(t_{l-1})\|^2 + 2k\int_{t_{l-1}}^{t_l}\mathrm{e}^{\mu_{t_{l-1}}(t-s)}Eb^2(s)\,\mathrm{d}s\right). \quad (13)$$

Hence, it follows from (12) to (13) that

$$\begin{aligned} E\|x(t)\|^2 \leqslant{}& \lambda_m^{-(j+1)}\mathrm{e}^{\mu_{t_0}(t_1-t_0)+\cdots+\mu_{t_j}(t-t_j)}E\|x_0\|^2 \\ &+ 2k\lambda_m^{-1}\int_{t_j}^{t}\mathrm{e}^{\mu_{t_j}(t-s)}Eb^2(s)\,\mathrm{d}s \\ &+ 2k\sum_{l=0}^{j-1}\lambda_m^{l-j-1}\int_{t_l}^{t_{l+1}}\mathrm{e}^{\mu_{t_l}(t_{l+1}-s)}Eb^2(s)\,\mathrm{d}s \\ &\times \mathrm{e}^{\mu_{t_{l+1}}(t_{l+2}-t_{l+1})+\cdots+\mu_{t_j}(t-t_j)}. \end{aligned} \quad (14)$$

Notice that $N_\sigma(t_0,t)=j$; $N_\sigma(s,t)=0, \forall s\in[t_j,t)$; and for $l=0,1,\ldots,j-1$, $N_\sigma(s,t)=j-l, \forall s\in[t_l,t_{l+1})$; $\lambda_M=1$ and $\lambda_m^{-1}=\mathrm{e}^h$ in (8). Then, we have $\lambda_m^{-1}=\mathrm{e}^{h[1+N_\sigma(t_j,t)]}$ for $s\in[t_j,t)$; and $\lambda_m^{-(j-l+1)}=\mathrm{e}^{h[1+N_\sigma(s,t)]}$ for $s\in[t_l,t_{l+1})$ ($l=0,1,\ldots,j-1$). In particular, we have $\lambda_m^{-(j+1)}=\mathrm{e}^{h[1+N_\sigma(t_0,t)]}$.

By Definition 4 and the definitions of a^-, a^+, λ_m and μ_{t_l}, one can get $\mu_{t_0}(t_1-t_0)+\mu_{t_1}(t_2-t_1)+\cdots+\mu_{t_j}(t-t_j)\leqslant\Sigma(t_0,t)$; $\mu_{t_j}(t-s)\leqslant\Sigma(s,t)$; and for $l=0,1,\ldots,j-1$, $\mu_{t_l}(t_{l+1}-s)+\mu_{t_{l+1}}(t_{l+2}-t_{l+1})+\cdots+\mu_{t_j}(t-t_j)\leqslant\Sigma(s,t)$, where $\Sigma(s,t)=(1/\lambda_m)[a^+T_\sigma^+(s,t;\mathscr{A})-a^-T_\sigma^-(s,t;\mathscr{A})+2kc^2(t-s)]$. This together with (14) gives $E\|x(t)\|^2\leqslant\mathrm{e}^{-h+hN_\sigma(t_0,t)+\Sigma(t_0,t)}E\|x_0\|^2 + 2k\mathrm{e}^{-h}\sum_{l=1}^{j}\int_{t_{l-1}}^{t_l}\mathrm{e}^{hN_\sigma(s,t)+\Sigma(s,t)}Eb^2(s)\,\mathrm{d}s + 2k\mathrm{e}^{-h}\int_{t_j}^{t}\mathrm{e}^{hN_\sigma(s,t)+\Sigma(s,t)}Eb^2(s)\,\mathrm{d}s$. Thus, by Remark 4 we have (11).

Theorem 1. *For System* (2), *if $E\|x_0\|^2<\infty$ and $\alpha(t,x)$ satisfies* (9) *with $c<\sqrt{a^-/(2k)}$, then*

(i) *when $b(t)$ satisfies $C\triangleq\sup_{t\geqslant t_0}Eb^2(t)<\infty$, System* (2) *is MS stable for any $a^*\in(2kc^2,a^-]$ and $\sigma(t)\in\mathscr{S}_2[a^*,h\lambda_m/(a^*-2kc^2);\mathscr{A}]$;*

(ii) *when $b(t)$ satisfies*

$$\int_{t_0}^{\infty}Eb^2(t)\,\mathrm{d}t<\infty. \quad (15)$$

System (2) *is MS stable for any $a^*\in(2kc^2,a^-]$ and $\sigma(t)\in\mathscr{S}_1[a^*,h\lambda_m/(a^*-2kc^2);\mathscr{A}]$. Here, a^-, λ_m, h are given by* (6)–(8).

Proof. For any given $x_0\in\mathbb{R}^n$, by Friedman (1975) there is a unique solution process $\{x(t),\forall t\geqslant t_0\}$ with $x(t_0)=x_0$. So, we need only to prove the MS stability.

(i) By the definition of $\Pi(\cdot,\cdot)$ in (11) and (3) we have

$$\Pi(t_0,t)\leqslant hN_0+\left(\frac{h}{\tau_\sigma}-\frac{a^*-2kc^2}{\lambda_m}\right)(t-t_0), \quad (16)$$

$$\Pi(s,t)\leqslant hN_0+\left(\frac{h}{\tau_\sigma}-\frac{a^*-2kc^2}{\lambda_m}\right)(t-s). \quad (17)$$

From Definition 4, $\sigma(t)\in\mathscr{S}_2[a^*,h\lambda_m/(a^*-2kc^2);\mathscr{A}]$ implies that $\tau_\sigma>h\lambda_m/(a^*-2kc^2)$, or equivalently, $h/\tau_\sigma-(a^*-2kc^2)/\lambda_m<0$. This together with (11) and $C=\sup_{t\geqslant t_0}Eb^2(t)<\infty$ gives $E\|x(t)\|^2\leqslant\mathrm{e}^{h+hN_0}(E\|x_0\|^2+2kC\int_{t_0}^{\infty}\mathrm{e}^{(h/\tau_\sigma-(a^*-2kc^2)/\lambda_m)s}\,\mathrm{d}s)<\infty$. Thus, System (2) is MS stable.

(ii) Noticing that for any $\sigma(t)\in\mathscr{S}_1[a^*,h\lambda_m/(a^*-2kc^2);\mathscr{A}]$ and $t-s\geqslant0$, we have (16)–(17) and $h/\tau_\sigma-(a^*-2kc^2)/\lambda_m\leqslant0$. This together with (11) and (15) leads to $E\|x(t)\|^2\leqslant\mathrm{e}^{h+hN_0}(E\|x_0\|^2+2k\int_{t_0}^{\infty}Eb^2(s)\,\mathrm{d}s)<\infty$. Thus, System (2) is MS stable. □

Remark 5. By Definition 4, $\sigma(t)\in\mathscr{S}_2[a^*,h\lambda_m/(a^*-2kc^2);\mathscr{A}]$ in Theorem 1 (i) implies that the average dwell-time τ_σ of $\sigma(t)$ satisfies the inequality $\tau_\sigma>h\lambda_m/(a^*-2kc^2)$, which gives a lower bound of τ_σ. This lower bound is weakened to $\tau_\sigma\geqslant h\lambda_m/(a^*-2kc^2)$ in Theorem 1 (ii). Such a condition seems unavoidable even for the switched deterministic system case:

Example 1. Consider system $\dot{x}(t)=A_{\sigma(t)}x(t)$ with $x(0)=\binom{1}{1}$, $\sigma(t):\ [0,\infty)\to\{1,2\}$, $A_1=\begin{pmatrix}-1 & 3\\ 0 & -1\end{pmatrix}$ and $A_2=A_1^{\mathrm{T}}$. By some simple calculations, one can see that $\lim_{t\to\infty}\|x(t)\|=\infty$, if

$$\sigma(t)=\begin{cases}1, & t\in[2l,2l+1),\\ 2, & t\in[2l+1,2l+2);\end{cases}$$

and $\lim_{t\to\infty}\|x(t)\|=0$, if

$$\sigma(t)=\begin{cases}1, & t\in[6l,6l+3),\\ 2, & t\in[6l+3,6l+6),\end{cases}$$

where $l=1,2,\ldots$.

This shows that for general switched systems, the whole systems may be unstable even if its all subsystems are stable, unless some limitations to lower bound of τ_σ of the switching law $\sigma(t)$ are imposed.

Theorem 2. *In addition to the conditions of Theorem* 1,

(i) *if the condition* (15) *on $b(t)$ is replaced by*

$$\int_{t_0}^{\infty}\exp\left(\frac{a^--2kc^2}{\lambda_m}t\right)Eb^2(t)\,\mathrm{d}t<\infty, \quad (18)$$

then for any $a^*\in(2kc^2,a^-]$ and $\sigma(t)\in\mathscr{S}_1[a^*,\tau^*;\mathscr{A}]$, System (2) *is EMS stable with convergent rate a. Here, $a\in(0,(a^*-2kc^2)/\lambda_m)$, $\tau^*=h/((a^*-2kc^2)/\lambda_m-a)$, and a^-, λ_m and h are given by* (6)–(8).

(ii) *if the condition on $\alpha(t,x(t))$ is strengthened to*

$$\|\alpha_r(t,x(t))\|\leqslant c\|x(t)\|,\quad r=1,2,\ldots,k, \quad (19)$$

where c is a nonnegative constant satisfying $kc^2<a^-$, then for any given $a^*\in(kc^2,a^-]$ and $\sigma(t)\in\mathscr{S}_1[a^*,\tau^*;\mathscr{A}]$, System (2) *is EMS stable with convergent rate a. Here, $a\in(0,(a^*-kc^2)/\lambda_m)$, $\tau^*=h/((a^*-kc^2)/\lambda_m-a)$, and a^-, λ_m, h are given by* (6)–(8).

Proof. The existence and uniqueness of $\{x(t), \forall t \geqslant t_0\}$ follows from Friedman (1975), so it suffices to show the EMS stability of System (2).

(i) Take arbitrarily $P_i \in \mathscr{P}_i$ $(i \in \mathscr{I})$. By Definition 4 and $\tau^* = h/((a^* - 2kc^2)/\lambda_m - a)$, for any $\sigma(t) \in \mathscr{S}_1[a^*, \tau^*; \mathscr{A}]$, we have $\tau_\sigma \geqslant \tau^*$, or equivalently, $a \leqslant (a^* - 2kc^2)/\lambda_m - h/\tau_\sigma$. This together with (3) gives $\Pi(s,t) \leqslant hN_0 - a(t-s)$. Then, from (11) and $0 < a < (a^* - 2kc^2)/\lambda_m < (a^- - 2kc^2)/\lambda_m$ it follows that $E\|x(t)\|^2 e^{-(1+N_0)h + a(t-t_0)} \leqslant E\|x_0\|^2 + 2k \int_{t_0}^{\infty} e^{a(s-t_0)} Eb^2(s)\,ds \leqslant E\|x_0\|^2 + 2k \int_{t_0}^{\infty} e^{((a^- - 2kc^2)/\lambda_m)(s-t_0)} Eb^2(s)\,ds$. Hence, by (18) and $a > 0$ we know that System (2) is EMS stable with convergent rate a.

(ii) For any given $t \geqslant t_0$, suppose that $\sigma(t)$ has j switching points in the time interval $[t_0, t)$, and denote these j switching points by $t_1, t_2, \ldots$ and t_j, respectively; and, for $l = 0, 1, \ldots, j$, let $\mu_{t_l} = [\lambda_M(A^{\mathrm{T}}_{\sigma(t_l)} P_{\sigma(t_l)} + P_{\sigma(t_l)} A_{\sigma(t_l)}) + kc^2]/\lambda_m(P_{\sigma(t_l)})$. Then, noticing that $\lambda_M(P_{\sigma(t_l)}) = 1$, by Corollary 1 we have $E\{x^{\mathrm{T}}(t) P_{\sigma(t_j)} x(t)\} \leqslant e^{\mu_{t_j}(t-t_j)} E\{x^{\mathrm{T}}(t_j) P_{\sigma(t_j)} x(t_j)\} \leqslant e^{\mu_{t_j}(t-t_j)} E\|x(t_j)\|^2$. Further, by the definition (7) of λ_m we have $E\|x(t)\|^2 \leqslant \lambda_m^{-1} e^{\mu_{t_j}(t-t_j)} E\|x(t_j)\|^2$. Similarly, we can get

$$
\begin{cases}
E\|x(t_j)\|^2 & \leqslant \quad \lambda_m^{-1} e^{\mu_{t_{j-1}}(t_j - t_{j-1})} E\|x(t_{j-1})\|^2, \\
 & \vdots \\
E\|x(t_1)\|^2 & \leqslant \quad \lambda_m^{-1} e^{\mu_{t_0}(t_1 - t_0)} E\|x(t_0)\|^2.
\end{cases}
$$

Hence, $E\|x(t)\|^2 \leqslant \lambda_m^{-(j+1)} e^{\mu_{t_0}(t_1-t_0)+\cdots+\mu_{t_j}(t-t_j)} E\|x_0\|^2$. By (8) and $\lambda_M = 1$ we have $\lambda_m^{-1} = e^h$, which together with $N_\sigma(t_0, t) = j$ gives $\lambda_m^{-(j+1)} = e^{h + hN_\sigma(t_0,t)}$.

From (4) to (7) it follows that $\lambda_M(A^{\mathrm{T}}_{\sigma(t_l)} P_{\sigma(t_l)} + P_{\sigma(t_l)} A_{\sigma(t_l)}) \leqslant a^+$ when $\Lambda(A_{\sigma(t_l)}) \subset \mathbb{C}^-$, and $\lambda_M(A^{\mathrm{T}}_{\sigma(t_l)} P_{\sigma(t_l)} + P_{\sigma(t_l)} A_{\sigma(t_l)}) \leqslant -a^-$ when $\Lambda(A_{\sigma(t_l)}) \not\subset \mathbb{C}^-$. Then, by Definition 4 we have $\mu_{t_0}(t_1 - t_0) + \mu_{t_1}(t_2 - t_1) + \cdots + \mu_{t_j}(t - t_j) \leqslant [a^+ T^+_\sigma(t_0, t; \mathscr{A}) - a^- T^-_\sigma(t_0, t; \mathscr{A}) + kc^2(t - t_0)]/\lambda_m$. This implies $E\|x(t)\|^2 \leqslant \exp(h + hN_\sigma(t_0,t) + kc^2(t-t_0)/\lambda_m + [a^+ T^+_\sigma(t_0,t;\mathscr{A}) - a^- T^-_\sigma(t_0,t;\mathscr{A})]/\lambda_m) E\|x_0\|^2$, which together with Remark 4 leads to

$$
\begin{aligned}
E\|x(t)\|^2 \leqslant \exp(h &+ hN_\sigma(t_0, t) \\
&+ (kc^2 - a^*)(t - t_0)/\lambda_m) E\|x_0\|^2. \qquad (20)
\end{aligned}
$$

By Definition 4 and

$$\tau^* = h/((a^* - kc^2)/\lambda_m - a)$$

we see that for $\forall \sigma(t) \in \mathscr{S}_1[a^*, \tau^*; \mathscr{A}]$, $\tau_\sigma \geqslant \tau^*$, or equivalently, $a \leqslant (a^* - kc^2)/\lambda_m - h/\tau_\sigma$. This together with (3) and (20) gives

$$E\|x(t)\|^2 \leqslant e^{h(1+N_0) - a(t-t_0)} E\|x_0\|^2.$$

Thus, System (2) is EMS stable with convergent rate a. $\square$

Remark 6. In fact, for any PD matrix $P_i (i \in \mathscr{I})$, if all the conditions of Theorems 1 and 2 hold with a^+, a^-, λ_m and kc^2 replaced by

$$\max_{1 \leqslant i \leqslant N} \max(\lambda_M(P_i A_i + A_i^{\mathrm{T}} P_i), 0),$$

$$\min_{1 \leqslant i \leqslant N} |\min(\lambda_M(P_i A_i + A_i^{\mathrm{T}} P_i), 0)|,$$

$\min_{1 \leqslant i \leqslant N} \lambda_m(P_i)$ and $kc^2 \max_{1 \leqslant i \leqslant N} \lambda_M(P_i)$, respectively, then the conclusions of Theorems 1–2 are still true.

All the above results have their corresponding corollaries. Here, we give only two corollaries to Theorem 2 (i).

Corollary 2. *For System* (2), *if* $A_i + A_i^{\mathrm{T}} < 0$ $(i \in \mathscr{I})$, *and* (18) *holds with* a^- *replaced by* $\mu \triangleq \min_{1 \leqslant i \leqslant N} |\lambda_M(A_i + A_i^{\mathrm{T}})|$, *then under the conditions of Theorem* 2(i), *System* (2) *is EMS stable with convergent rate* $\mu - 2kc^2$ *for all* $\sigma(t)$.

Proof. Noticing that the definition of μ, and for this corollary, since $A_i + A_i^{\mathrm{T}} < 0$ we have $T^+_\sigma(s, t; \mathscr{A}) \equiv 0$, $a^+ = 0$ and $a^- = \mu$, and can take $P_i \equiv I (i \in \mathscr{I})$ which implies $\lambda_M = \lambda_m = 1$. Similar to the proof of Lemma 2, we know System (2) is EMS stable with convergent rate $\mu - 2kc^2$. $\square$

Corollary 3. *For System* (2), *in addition to the conditions of Theorem* 2(i), *suppose* $\Lambda(A_i) \subset \mathbb{C}^- (i \in \mathscr{I})$, *and* $A^{\mathrm{T}}_{i_p} + A_{i_p} < 0$ *for* $p = 1, \ldots, N_1 (N_1 < N)$. *Let*

$$\mu = \min\left(\min_{1 \leqslant p \leqslant N_1} |\lambda_M(A_{i_p} + A^{\mathrm{T}}_{i_p})|, \min_{N_1+1 \leqslant p \leqslant N} |a_{i_p}| \right)$$

with a_i *given in* (4). *Then the conclusion of Theorem* 2(i) *is true for any* $a \in (0, (\mu - 2kc^2)/\min(\lambda_m, 1))$ *and* $\sigma(t)$, *provided that* (18) *holds with* a^- *replaced by* μ, *and the average dwell-time* τ_σ *on* $\{A_{i_{N_1+1}}, \ldots, A_{i_N}\}$ *satisfies* $\tau_\sigma \geqslant h/((\mu - 2kc^2)/\min(\lambda_m, 1) - a)$.

Remark 7. Corollary 2 shows that when A_i $(i \in \mathscr{I})$ have a common Lyapunov matrix $P = I$, there is no need to make any restriction to $\sigma(t)$ to ensure the EMS stability of the system, namely the switch can be arbitrary. This remark is also valid when $P \neq I$.

Remark 8. Corollary 3 shows that for any $\sigma(t)$ there is no need to impose any restriction to the switching frequency of $\sigma(t)$ on $\{A_{i_1}, \ldots, A_{i_{N_1}}\}$.

4. Stabilization control of SS systems

Based on Theorems 1–2, we can establish a series of stabilization results. Here, for simplicity, we will discuss only the ones corresponding to Theorem 2(ii), that is, under the assumption (19), we would like to design a state feedback control of the form $u_{\sigma(t)}(t) = K_{\sigma(t)} x(t)$ such that the closed-loop system

$$dx(t) = \bar{A}_{\sigma(t)} x(t)\,dt + \alpha(t, x(t))\,dW(t), \quad x(t_0) = x_0 \qquad (21)$$

is EMS stable, where $\bar{A}_{\sigma(t)} = A_{\sigma(t)} + B_{\sigma(t)} K_{\sigma(t)}$.

In the following, we denote all K_i by K with proper dimension, for short.

Let $\bar{\mathscr{A}} = \{\bar{A}_i, i \in \mathscr{I}\}$. Obviously, if all $\bar{A}_i$ in $\bar{\mathscr{A}}$ are stable, then for any given PD matrix Q_i, there exists a PD matrix $P_i(K)$ such that

$$\bar{A}_i^{\mathrm{T}} P_i(K) + P_i(K)\bar{A}_i = -Q_i \quad \forall i \in \mathscr{I}. \tag{22}$$

By Lemma 3.2 of Cheng et al. (2005), for a controllable subsystem (A_i, B_i) of System (1), one can arbitrarily change the $\lambda_M(P_i(K))$ with proper feedback matrix K; but for an UCbS subsystem (A_i, B_i) of System (1), the case is different. This means that one has to explicitly characterize the dominant impact of the uncontrollable eigenvalue of (A_i, B_i) on $\lambda_M(P_i(K))$. To do so, set

$$\mathscr{K}(A_i, B_i) = \{K : \ v(\bar{A}_i, B_i) \leqslant \mu(A_i, B_i)\}, \tag{23}$$

where $v(\bar{A}_i, B_i)$ is the largest real part of all controllable eigenvalues of (A_i, B_i), and $\mu(A_i, B_i)$ denotes the largest negative real part of all uncontrollable eigenvalues of (A_i, B_i).

Then, for an UCbS subsystem (A_i, B_i), we define

$$\begin{cases} \eta_M^{(i)} = \min\limits_{K \in \mathscr{K}(A_i, B_i)} \lambda_M(P_i(K)), \\ \mathscr{K}_{0i} = \{K : K \in \mathscr{K}(A_i, B_i), \eta_M^{(i)} = \lambda_M(P_i(K))\}. \end{cases} \tag{24}$$

Further, for System (1), suppose that subsystems (A_{i_p}, B_{i_p}) are controllable for $p = 1, \ldots, N_1(\leqslant N)$, and are UCbS for $p = N_1 + 1, \ldots, N$. For the PD matrices Q_i and $P_i(K)$ in (22), let $\Psi_i = \lambda_M(\bar{A}_i^{\mathrm{T}} P_i(K) + P_i(K)\bar{A}_i)$; and similar to (4)–(6), introduce

$$\begin{cases} a^+(\bar{\mathscr{A}}) = \max\limits_{i \in \mathscr{I}} \max(\Psi_i, 0), \\ a^-(\bar{\mathscr{A}}) = \min\limits_{i \in \mathscr{I}} |\min(\Psi_i, 0)|, \end{cases} \tag{25}$$

$$\begin{cases} \eta_M = \max\left(\max\limits_{1 \leqslant p \leqslant N_1} \lambda_M(P_{i_p}(K)), \max\limits_{N_1+1 \leqslant p \leqslant N} \eta_M^{(i_p)} \right), \\ \eta_m = \min\Bigg(\min\limits_{1 \leqslant p \leqslant N_1} \lambda_m(P_{i_p}(K)), \\ \qquad \min\limits_{N_1+1 \leqslant p \leqslant N} \inf\limits_{K \in \mathscr{K}_{0i_p}} \lambda_m(P_{i_p}(K)) \Bigg). \end{cases} \tag{26}$$

Remark 9. Obviously, for any UCbS subsystem (A_i, B_i) the set $\mathscr{K}_{0i}$ is nonempty.

Remark 10. For any given PD matrices Q_i, it is clear that $a^+(\bar{\mathscr{A}}) \equiv 0$ for any K guaranteeing all $\bar{A}_i$ stable. Meanwhile, by Definition 4 we have $T_\sigma^+(s, t; \bar{\mathscr{A}}) \equiv 0$, so a^* can be taken as $a^-(\bar{\mathscr{A}})$ in both $\mathscr{S}_1[a^*, \tau^*; \bar{\mathscr{A}}]$ and $\mathscr{S}_2[a^*, \tau^*; \bar{\mathscr{A}}]$.

Remark 11. When $N_1 < N$, the constant η_M has a lower bound $\max_{N_1+1 \leqslant p \leqslant N} \eta_M^{(i_p)}$.

Replacing a^- and kc^2 with $a^-(\bar{\mathscr{A}})$ and $kc^2\eta_M$, respectively, in Theorem 2(ii), by Remark 5 we can get

Lemma 3. *If $\alpha(t, x)$ satisfies (19), and there are K_i $(i \in \mathscr{I})$ such that all the $\bar{A}_i = A_i + B_iK_i \in \bar{\mathscr{A}}$ $(i \in \mathscr{I})$ are stable and the PD matrices $P_i(K_i)$ and Q_i $(i \in \mathscr{I})$ satisfy*

$$kc^2\eta_M < a^-(\bar{\mathscr{A}}), \tag{27}$$

then for any $\sigma(t) \in \mathscr{S}_1[a^-(\bar{\mathscr{A}}), \tau_a; \bar{\mathscr{A}}]$, System (21) is EMS stable with convergent rate a. Here, $a \in (0, (a^-(\bar{\mathscr{A}}) - kc^2\eta_M)/\eta_m]$, $\tau_a = (\ln(\eta_M/\eta_m))/((a^-(\bar{\mathscr{A}}) - kc^2\eta_M)/\eta_m - a)$, and $a^-(\bar{\mathscr{A}})$, η_M and η_m are given in (25)–(26).

Remark 12. Stabilizing System (21) is not essentially dependent on the choice of PD matrices Q_i, although the definitions of $a^-(\bar{\mathscr{A}})$, η_M and η_m in Lemma 3 and (25)–(26) are obviously related to the choice of Q_i. In particular, when $\eta_M = \eta_m$ (denote them with η, for short), Lemma 3 becomes a special case of Corollary 1 (with some changes on the conditions and conclusions in accordance with Theorem 2(ii)): *for any given switching law $\sigma(t)$, System (21) is EMS stable with convergent rate $a^-(\bar{\mathscr{A}})/\eta - kc^2$.*

Definition 5. For $\forall t \geqslant t_0$, if the value of $\sigma(s)$ is well-defined and available to the control design for all $s \in [t_0, t]$, then the process $\sigma(\cdot)$ is said observable.

Remark 13. By Definition 5, if a switching law $\sigma(\cdot)$ is observable, then, at any time instant $t > t_0$, the value $\sigma(s)$, $\forall s \in [t_0, t]$, is given and available to control design, although the information on $\{\sigma(s) : \ s > t\}$ may be unknown. This is the essential difference between the switched system control and the conventional time-varying system control (Anderson & Moore, 1971). The latter requires that the value of the time process $\sigma(t)$ (and hence, $A_{\sigma(t)}$ and $B_{\sigma(t)}$) on the whole control time interval $[t_0, \infty)$ is completely known at time t_0.

Theorem 3. *For System (1), suppose that subsystems (A_i, B_i) $(i \in \mathscr{I})$ are controllable (that is, $N_1 = N$ in (26)), $\alpha(t, x(t))$ satisfies (19), switching law $\sigma(\cdot)$ is observable and $\sigma(\cdot) \in \mathscr{S}[\tau^*, N_0]$ for some given constants $N_0 \geqslant 0$ and $\tau^* > 0$, then there exists a state feedback $u_{\sigma(t)}(t) = K_{\sigma(t)}x(t)$ such that System (21) is EMS stable with convergent rate $a \in (0, [1 - kc^2\eta_M]/\eta_m)$. Here, $\mathscr{S}[\tau^*, N_0]$, N_0 and τ^* are given in Definition 3, η_m and η_M in (26).*

Proof. By Remark 12, one can simply take $Q_i = I$ $(i \in \mathscr{I})$. In this case, we have $a^-(\bar{\mathscr{A}}) = 1$.

Since (A_i, B_i) are controllable, there exist feedback matrices $K_i \in \mathbb{R}^{m_i \times n}$ such that $\Lambda(\bar{A}_i) \subset (-\infty, 0)$, where $\bar{A}_i = A_i + B_iK_i$. By Lemma 3.2 of Cheng et al. (2005) we can obtain $\lambda_M(P_i) = \|P_i\| = \|\int_0^\infty \mathrm{e}^{\bar{A}_i^{\mathrm{T}} t}\mathrm{e}^{\bar{A}_i t}\, \mathrm{d}t\| \leqslant (M^2/2)\lambda^{n^2+n-3}$. Thus, without loss of generality, we assume that K_i has already been chosen such that η_m and η_M in (26) are sufficiently small so that

$$kc^2\eta_M + \frac{\eta_m}{\tau^*} \ln \frac{\eta_M}{\eta_m} < 1 = a^-. \tag{28}$$

In fact, noticing $\lim_{\eta\to 0+}\eta\ln\eta=0$, from the straightforward inequality $|(\eta_m/\tau^*)\ln(\eta_M/\eta_m)|\leqslant(1/\tau^*)[|\eta_m\ln\eta_m|+|\eta_M\ln\eta_M|]$ we know that when η_m and η_M are sufficiently small, (28) must be true.

In this case, if $\eta_m=\eta_M=\eta$, then by Remark 12, System (21) is EMS stable with convergent rate $1/\eta-kc^2$; when $\eta_m\neq\eta_M$, let $a=(1-kc^2\eta_M)/\eta_m-(1/\tau^*)\ln(\eta_M/\eta_m)$. Then, by (28) one can see that $a\in(0,[1-kc^2\eta_M]/\eta_m)$ and the average dwell-time τ^* of $\mathscr{S}[\tau^*,N_0]$ can be expressed as $\tau^*=(\ln(\eta_M/\eta_m))/((1-kc^2\eta_M)/\eta_m-a)$. Hence, by Definitions 3–4, we have $\sigma(\cdot)\in\mathscr{S}_1[1,\tau^*;\bar{\mathscr{A}}]$. Thus, by Lemma 3, System (21) is EMS stable with convergent rate a. $\square$

Remark 14. By Theorem 3, for any given constant $c>0$ in (19), System (21) can be stabilized by state feedback control if all (A_i,B_i) $(i\in\mathscr{I})$ are controllable. However, when some subsystem is uncontrollable, in order to stabilize System (1), some restriction to the constant c in (19) seems necessary. To see this, we present the following example.

Example 2. Assume (A,B) is a subsystem of a switched system with

$$A=\begin{pmatrix}-2 & 0\\ 0 & -1\end{pmatrix}\quad\text{and}\quad B=\begin{pmatrix}1\\0\end{pmatrix}.$$

Obviously, (A,B) is uncontrollable. For any given feedback matrix $K=(k_1,k_2)$, the state matrix of the closed-loop system is with the form

$$\bar{A}=A+BK=\begin{pmatrix}k_1-2 & k_2\\ 0 & -1\end{pmatrix},$$

which gives

$$\mathrm{e}^{\bar{A}t}=\begin{pmatrix}\mathrm{e}^{(k_1-2)t} & *\\ 0 & \mathrm{e}^{-t}\end{pmatrix},$$

where $*$ denotes a function of t depending on k_1 and k_2. When K is such that $\Lambda(\bar{A})\subset(-\infty,0)$ (in this case, $k_1<2$ and k_2 can be an arbitrary real number), the solution matrix P of the Lyapunov equation $\bar{A}^{\mathrm{T}}P+P\bar{A}=-I$ can be expressed as

$$P=\int_0^\infty \mathrm{e}^{A^{\mathrm{T}}t}\mathrm{e}^{At}\,\mathrm{d}t=\begin{pmatrix}-\dfrac{1}{2(k_1-2)} & \star\\ \star & \dfrac{1}{2}+\int_0^\infty *^2\,\mathrm{d}t\end{pmatrix},$$

where $\star$ is a nonnegative constant depending on k_1 and k_2. Hence, we have $\lambda_M(P)=\|P\|\geqslant\frac{1}{2}+\int_0^\infty *^2\,\mathrm{d}t\geqslant\frac{1}{2}$. This implies that the η_M in (26) is greater than $\frac{1}{2}$ for all state feedback controls. Thus, in order to get the condition $1-kc^2\eta_M>0$, c must be less than $1/\sqrt{k\eta_M}$, where k is the dimension of noise coefficient matrix $\alpha(t,x(t))$.

For the case where only a part of subsystems of the System (1) is controllable, we have the following stabilization theorem.

Theorem 4. *For System* (1), *suppose that subsystems* (A_{i_p}, B_{i_p}) *are controllable for* $p=1,\dots,N_1(<N)$, *and UCbS for* $p=N_1+1,\dots,N$, $\alpha(t,x(t))$ *satisfies* (19), *switching law* $\sigma(t)$ *is observable and* $\sigma(t)\in\mathscr{S}[\tau^*,N_0]$ *for some constants* $N_0\geqslant 0$ *and* $\tau^*>0$. *If* $1-kc^2\max_{N_1+1\leqslant p\leqslant N}\eta_M^{(i_p)}>0$, *then there exists a state feedback* $u_{\sigma(t)}(t)=K_{\sigma(t)}x(t)$ *such that System* (21) *is EMS stable with convergent rate* $a\in(0,[1-kc^2\eta_M]/\eta_m)$. *Here,* $\mathscr{S}[\cdot,\cdot]$ *and* $\eta_M^{(i_p)}$ *is given in Definition* 3 *and* (24), η_m *and* η_M *in* (26).

Proof. Similar to Theorem 3, for (22), choose $Q_i=I$ $(i\in\mathscr{I})$. Then, $a^-(\bar{\mathscr{A}})=1$. Without loss of generality, assume $i_p=p$. Then, (A_i,B_i) is controllable for $i=1,\dots,N_1(<N)$ and UCbS for $i=N_1+1,\dots,N$.

Since (A_i,B_i) $(N_1+1\leqslant i\leqslant N)$ is stabilizable, there exists $K_i\in\mathscr{K}_0(A_i,B_i)$ such that the solution $P_i(K_i)$ of (22) satisfies $\eta_M^{(i)}=\lambda_M(P_i(K_i))$. This together with the condition of this theorem implies that $1-kc^2\max_{N_1+1\leqslant i\leqslant N}\eta_M^{(i)}>0$.

Notice that (A_i,B_i) $(1\leqslant i\leqslant N_1)$ are controllable. Then, similar to the proof of Theorem 3, by Lemma 3.2 of Cheng et al. (2005) we know that there exists $K_i\in\mathbb{R}^{m_i\times n}$ such that not only (22) has a unique PD solution matrix $P_i(K_i)$ but also $\lambda_M(P_i(K_i))$ is sufficiently small so that $\max_{1\leqslant i\leqslant N_1}\lambda_M(P_i(K_i))\leqslant\max_{N_1+1\leqslant i\leqslant N}\eta_M^{(i)}$, and $\min_{1\leqslant i\leqslant N_1}\lambda_m(P_i(K_i))\leqslant\min_{N_1+1\leqslant i\leqslant N}\inf_{K\in\mathscr{K}_{0i}}\lambda_m(P_i(K))$, or equivalently, by (26),

$$\eta_M=\max_{N_1+1\leqslant i\leqslant N}\eta_M^{(i)},\quad \eta_m=\min_{1\leqslant i\leqslant N_1}\lambda_m(P_i(K_i)).\tag{29}$$

As in Theorem 3, we can choose K_i such that η_m is sufficiently small so that (28) holds, where η_M and η_m are given in (29).

Let $a=(1-kc^2\eta_M)/\eta_m-(1/\tau^*)\ln(\eta_M/\eta_m)$. Then $a>0$ and the average dwell-time τ^* of $\mathscr{S}[\tau^*,N_0]$ can be expressed as $\tau^*=((1-kc^2\eta_M)/\eta_m-a)^{-1}\ln(\eta_M/\eta_m)$. Hence, by Definitions 3–4, we have $\sigma(t)\in\mathscr{S}_1[a^-(\bar{\mathscr{A}}),\tau^*;\bar{\mathscr{A}}]$, where $a^-(\bar{\mathscr{A}})=1$. Therefore, by Lemma 3, System (21) is EMS stable with convergent rate a. $\square$

5. Concluding remark

By introducing two compact matrix sets $\mathscr{D}$ and $\mathscr{D}_\alpha$, we have studied the MS stability of a class of switched system with stochastic disturbances. Based on the concept of average dwell-time and the ratio of the total time running on all unstable subsystems to the total time running on all stable subsystems, some sufficient conditions on MS stability and EMS stability are given, respectively. The impact of the uncontrollable part of an UCbS stochastic system on the convergent rate of the closed-loop system is analyzed, and the stabilization control problems of two classes of SS control systems are investigated. One is with all subsystems controllable, and the other is with some subsystems controllable and some UCbS. Some sufficient conditions on the EMS stability of the closed-loop systems, and the convergent rates of the closed-loop systems are given. Unlike conventional time-varying parameter systems, here it is only required that the switching law is observable.

References

Anderson, B. D. O., & Moore, J. B. (1971). *Linear optimal control*. Englewood Cliffs: Prentice-Hall.

Branicky, M. S. (1998). Multiple Lyapunov functions and other analysis tools for switched and hybrid systems. *IEEE Transactions on Automatic Control*, *43*(4), 475–482.

Caines, P. E., & Zhang, J. F. (1996). On the adaptive control of jump parameter systems via nonlinear filtering. *SIAM Journal of Control and Optimization*, *33*(6), 1758–1777.

Cheng, D., Guo, L., Lin, Y., & Wang, Y. (2005). Stabilization of switched linear systems. *IEEE Transactions on Automatic Control*, *50*(5), 661–666.

Daafouz, J., Riedinger, P., & Iung, C. (2002). Stability analysis and control synthesis for switched systems: A switched Lyapunov function approach. *IEEE Transactions on Automatic Control*, *47*(11), 1883–1887.

Friedman, A. (1975). *Stochastic differential equations and applications* (Vol. 1). New York: Academic Press.

Giua, A., Seatzu, C., & Van Der Mee, C. (2001). Optimal control of switched autonomous linear Systems. *Proceedings of the 40th CDC* (pp. 2472–2477). Orlando, USA.

Hespanha, J. P. (2004). Uniform stability of switched linear systems: Extensions of LaSalle's invariance principle. *IEEE Transactions on Automatic Control*, *49*(4), 470–482.

Hespanha, J. P., & Morse, A. S. (1999). Stability of switched systems with average dwell-time. *Proceedings of the 38th CDC* (pp. 2655–2660) Phoenix, USA.

Ishii, H., & Francis, B. A. (2002). Stabilizing a linear system by switching control with dwell time. *IEEE Transactions on Automatic Control*, *47*(12), 1962–1973.

Li, Z. G., Wen, C. Y., & Soh, Y. C. (2001). Stabilization of a class of switched systems via designing switching laws. *IEEE Transactions on Automatic Control*, *46*(4), 665–670.

Liberzon, D., Hespanha, J. P., & Morse, A. S. (1999). Stability of switched systems: A Lie-algebraic condition. *Systems and Control Letters*, *37*(3), 117–122.

Liberzon, D., & Morse, A. S. (1999). Basic problem in stability and design of switched systems. *IEEE Control Systems*, *19*(5), 59–70.

Mariton, M. (1990). *Jump linear systems in automatic control*. New York: Marcel Dekker.

Sun, Z., Ge, S. S., & Lee, T. H. (2002). Controllability and reachability criteria for switched linear systems. *Automatica*, *38*(5), 775–786.

Xie, G., & Wang, L. (2003). Controllability and stabilizability of switched linear-systems. *Systems and Control Letters*, *48*(2), 135–155.

Zhai, G., Hu, B., Yasuda, K., & Michel, A. N. (2001). Stability analysis of switched systems with stable and unstable subsystems: an average dwell time approach. *International Journal of System Science*, *32*(8), 1055–1061.

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