Brief paper

Stability analysis and stabilization control of multi-variable switched stochastic systems

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Abstract

In this paper, the mean square (MS) stability and exponential mean square (EMS) stability of multi-variable switched stochastic systems are investigated. Based on the concept of the average dwell-time and the ratio of the total time running on all unstable subsystems to the total time running on all stable subsystems, some sufficient conditions are given to ensure the MS stability and EMS stability of the switched stochastic systems involved. Further, for the switched stochastic control systems with all subsystems controllable or stabilizable, EMS stabilization controls and sufficient conditions on EMS stabilization are presented, and the convergent rates of the closed-loop systems are obtained.

Keywords: Switched system; Stochastic system; Average dwell-time; Mean square stability; Stabilization control

1. Introduction

Since switched control systems exist widely in engineering technology and social systems (Mariton, 1990), the analysis and control problems of them have attracted extensive attention from many researchers. Many important progress and remarkable achievements have been made on issues such as controllability, reachability and stabilizability (Sun, Ge, & Lee, 2002; Xie & Wang, 2003), control and switching law design (Cheng, Guo, Lin, & Wang, 2005; Daafouz, Riedinger, & Jung, 2002; Ishii & Francis, 2002; Liberzon & Morse, 1999; Li, Wen, & Soh, 2001), optimal control (Giua, Seatzu, & Van Der Mee, 2001) and so on. Among others, stability analysis and stabilization control are two hot topics. The basic problems considered include stability analysis for systems with specific switching laws (Branicky, 1998), stability analysis for systems with arbitrary switching laws (Liberzon, Hespanha, & Morse, 1999), and design of stabilization switching laws (Daafouz et al., 2002; Li et al., 2001), etc. For the first topic, the concepts such as dwell-time and average dwell-time, etc. are introduced by Hespanha and Morse (1999), and then, used in Zhai, Hu, Yasaki, and Michel (2001) and Hespanha (2004), which make it possible for us to analyze the stability properties of switched systems with stable and unstable subsystems. For the second topic, common Lyapunov function and linear matrix inequality were used for stabilization control of linear switched systems with arbitrary switching laws (Liberzon et al., 1999). And the concepts of switching frequency and dwell-time were used for stabilization control of switched systems with controllable and uncontrollable subsystems in Cheng et al. (2005), where a subtle capacity characterization of feedback matrix on improving the convergent rate of the closed-loop linear time-invariant systems is provided, which is the key to our success of stabilization control for switched stochastic (SS) systems (see Section 4). The works mentioned above mainly focused on deterministic systems. For stochastic switched control systems, although some results have been given (Caines & Zhang, 1996, and the references therein), they are merely concentrated on the case where the switching laws are random, Markovian processes with known transition probability.
The systems studied in this paper are with deterministic switching laws and unknown stochastic disturbances. By describing quantitatively the quadratic stability and instability of matrices, based on the concept of the average dwell-time and the ratio of the total time running on all unstable subsystems to the total time running on all stable subsystems, some sufficient conditions on mean square (MS) stability and exponential mean square (EMS) stability are given, respectively. In addition, by analyzing the impact of the uncontrollable part of an uncontrollable but stabilizable (UCbS) stochastic system on the convergent rate of the closed-loop system, the stabilization control problems are investigated for two classes of SS control systems, one is with all subsystems controllable, and the other is with some subsystems controllable and some UCbS. Some sufficient conditions ensuring the EMS stability of the closed-loop systems are obtained, and the convergent rates of the closed-loop systems are analyzed. Unlike conventional time-varying parameter systems, here it is only required that the switching law $\sigma(t)$ is observable. In other words, for control designers, at any time instance $t > t_0$, only $\sigma(s), s \in [0, t]$ is known and available for control design. As for $\sigma(s)$ for $s \in (t, \infty)$, the designers probably do not know any information before the time instance $t$.

The remainder of this paper is organized as follows. In Section 2, some notations and definitions are introduced and the problems to be studied are formulated. In Section 3, the stability of autonomous (i.e. control-free) SS systems is analyzed, and some sufficient conditions on MS stability and EMS stability are given, respectively. In Section 4, the stabilization problem of SS systems is investigated, some sufficient conditions on EMS stability and the convergent rate of the closed-loop systems are obtained. Section 5 concludes the paper.

2. Notations and problem formulation

Consider the following SS system

$$
\dot{x}(t) = A_{\sigma(t)}(t)x(t) dt + B_{\sigma(t)}u_{\sigma(t)}(t) dt + \sigma(t, x(t)) dW(t)
$$

with initial condition $x(t_0) = x_0$, where function $\sigma(\cdot); [t_0, \infty) \rightarrow \mathcal{S} = \{1, 2, \ldots, N\}$ is the switching law and is deterministic (namely, not random), piecewise constant and right continuous; $A_i \in \mathbb{R}^{n \times n}$ and $B_i \in \mathbb{R}^{n \times m}$ for $i \in \mathcal{S}$ are constant matrices; $x(t) \in \mathbb{R}^n$, $u(t) \in \mathbb{R}^m$ ($i \in \mathcal{S}$) are system state and inputs, respectively; $\sigma(t, x) = (\sigma_1(t, x), \ldots, \sigma_N(t, x))^T \in \mathbb{R}^{n \times k}$ is the noise coefficient, continuous on $(t, x)$ and uniformly Lipschitz continuous on $x$; $W(t) \in \mathbb{R}^k$ is a standard Brownian motion.

Throughout the paper, $\mathbb{R}^{n \times m}$ denotes the set of all $n \times m$ real matrices; $E(\cdot)$ denotes the mathematical expectation; $\| \cdot \|$ denotes the Euclidean norm in the $n$-dimensional real space $\mathbb{R}^n$ and the corresponding induced matrix norm; $A(A)$ denotes the set of all eigenvalues of a square matrix $A$, and for a real symmetric matrix $A$, $\lambda_M(A)$ and $\lambda_m(A)$ denotes its maximal eigenvalue and minimal eigenvalue, respectively; $\mathbb{C}^-$ denotes the open left half complex plane; $I$ denotes the identity matrix with proper dimension; for any given matrix or vector $X$, $X^T$ denotes its transpose.

This paper investigates the MS stabilization control of the system (1). To do so, we introduce some concepts on MS stability for the following stochastic systems:

$$
\dot{x}(t) = A_{\sigma(t)}x(t) dt + \sigma(t, x(t)) dW(t), \quad x(t_0) = x_0.
$$

**Definition 1.** For System (2), if its solution process $\{x(t), t \geq t_0\}$ with $x(t_0) = x_0$ is existent and unique, and satisfies $\sup_{t \geq t_0} E \|x(t)\|^2 < \infty$, then the solution process is said MS stable; in addition, if for some positive constants $\alpha$ and $a$, $E \|x(t)\|^2 \leq M_0 e^{-\alpha(t-t_0)}$, $\forall t \geq t_0$, then the solution process is said EMS stable (with convergent rate $a$).

System (2) is said MS stable if for $\forall x_0 \in \mathbb{R}^n$, the $\{x(t), t \geq t_0\}$ with $x(t_0) = x_0$ is MS stable, and is said EMS stable if for $\forall x_0 \in \mathbb{R}^n$, the $\{x(t), t \geq t_0\}$ with $x(t_0) = x_0$ is EMS stable.

**Definition 2.** The problem to design a control law $u_{\sigma(t)}(t)$ such that System (1) is MS (or EMS) stable is called MS (correspondingly, EMS) stabilization control problem.

**Definition 3 (Hespanha & Morse, 1999).** For any given $\sigma(t)$ and $t > s \geq t_0$, let $N_\sigma(s, t)$ be the switching number of $\sigma(t)$ on interval $[s, t]$. For any given $N_0 > 0$ and $\tau^* > 0$, set $\mathcal{S}^* = \{\sigma(\cdot) : N_\sigma(s, t) \leq N_0 + (t - s)/\tau^*, \forall t \geq t_0, \forall \sigma(\cdot) \in [0, t_0]\}$. Then $\tau^*$ and $N_0$ are called average dwell-time and chatter bound of $\mathcal{S}^*$, respectively. For a given $\sigma(t) \in \mathcal{S}^*$, $N_0$, the constant $\tau_\sigma$ determined by $1/\tau_\sigma = \sup_{\tau \geq \tau^*} \sup_{t \geq t_0} (N_\sigma(s, t) - N_0)/(t - s)$ is called the average dwell-time of $\sigma(t)$.

**Remark 1.** In fact, for any given $N_0 > 0$, the average dwell-time $\tau^* > 0$ in Definition 3 is the minimum of the average dwell-time $\tau_\sigma$ of all $\sigma(t) \in \mathcal{S}^* = \{\sigma(\cdot) \in \mathcal{S}_{[t_0, t_0]} \}$ and we have

$$
N_\sigma(s, t) \leq N_0 + (t - s)/\tau_\sigma.
$$

3. Mean square stability of SS systems

For studying the MS and EMS stability of System (2), we first characterize the quadratically stable convergence rate of stable matrices.

For any given $n \times n$ positive definite (PD) matrix $P_0$, if $A_1^T P_0 + P_0 A_1 < 0$, then along the trajectory of $x = A_1 x$ Lyapunov function $V(x) = x^T P_0 x$ satisfies $\dot{V} = x^T [A_1^T P_0 + P_0 A_1] x \leq \gamma^2 V(x)$, where $\gamma = \sqrt{\lambda_M(A_1^T P_0 + P_0 A_1)} > 0, \lambda_M(P_0) > 0$. This implies that $V(x) \leq V(x(t_0)) e^{-\gamma(t-t_0)}, \forall t \geq t_0$. Here we name this $\gamma$ as the convergent rate of quadratic stability of $A_1$.

Inspired by this, we determine the largest $\gamma$ with respect to all PD matrices $P$. For any given $x \in (0, 1), \exists \mathcal{D} = \{P \in \mathbb{R}^{n \times n} : P \succeq 0, \|P\| = 1\}, \exists \mathcal{D}_\gamma = \{P \in \mathbb{R}^{n \times n} : \quad \|P\| = 1\}$. For any given $n \times n$ matrix set $\mathcal{S} = \{A_i, i \in \mathcal{S}\}$, let

$$
\alpha_i(\mathcal{S}) = \left\{ \begin{array}{ll}
\min_{P \in \mathcal{D}} \lambda_M(P A_i + A_i^T P), & A_i(\mathcal{S}) \subset \mathbb{C}^-,
\max_{P \in \mathcal{D}_\gamma} \lambda_M(P A_i + A_i^T P), & A_i(\mathcal{S}) \not\subset \mathbb{C}^{-};
\end{array} \right.
$$

For any given $n \times n$ matrix set $\mathcal{S} = \{A_i, i \in \mathcal{S}\}$, let

$$
\alpha_i(\mathcal{S}) = \left\{ \begin{array}{ll}
\min_{P \in \mathcal{D}} \lambda_M(P A_i + A_i^T P), & A_i(\mathcal{S}) \subset \mathbb{C}^-,
\max_{P \in \mathcal{D}_\gamma} \lambda_M(P A_i + A_i^T P), & A_i(\mathcal{S}) \not\subset \mathbb{C}^{-};
\end{array} \right.
$$
for any given stable matrix $A_i \in S$, let $\mathcal{P}_1(S) = \{P \in \mathcal{D}: \lambda_1(PA_i + A_i^T P) = \sigma_i(S)\}$, for any given unstable matrix $A_i \in S$, let $\mathcal{P}_2(S) = \{P \in \mathcal{D}_2: \lambda_1(PA_i + A_i^T P) = \sigma_i(S)\}$. Denote
\[ \mathcal{P}_i(S) = \begin{cases} \mathcal{P}_1(S), & \lambda_1(A_i) \subset \mathcal{C}^-; \\ \mathcal{P}_2(S), & \lambda_1(A_i) \subset \mathcal{C}; \end{cases} \]
(5)
\[
\begin{align*}
& a_+(S) = \max_{i \in I} \sigma_i(S), \quad 0; \\
& a_-(S) = \min_{i \in I} |\sigma_i(S)|, \quad 0; \\
& \lambda_m = \min_{i \in I} \inf_{P \in \mathcal{P}_i(S)} \lambda_m(P), \min_{P \in \mathcal{D}_2} \lambda_m(P), \quad h = \ln \frac{\lambda_M}{\lambda_m}; \\
& \lambda_M = \max_{i \in I} \sup_{P \in \mathcal{P}_i(S)} \lambda_m(P), \quad h = \ln \frac{\lambda_M}{\lambda_m}. 
\end{align*}
\]
(6)
(7)
(8)

In the following, we will denote $a_1(S), \mathcal{P}_1(S), a_+(S)$ and $a_-(S)$ by $a_i, \mathcal{P}_i, a_+^i$ and $a_-^i$ for short, respectively.

**Remark 2.** All sets $\mathcal{P}_i(i \in I)$ are nonempty, since $\mathcal{D}$ and $\mathcal{D}_2$ are bounded and closed, and $\lambda_2(PA_i + A_i^T P)$ is continuous and homogeneous on $P$. Besides, it is easy to show that for $\forall P_i \in \mathcal{P}_i$, $\lambda_M = 1, P_i A_i + A_i^T P_i \leq a_i I$.

**Remark 3.** When $A(A_i) \subset \mathcal{C}^-$, the $a_i$ defined on $\mathcal{D}$ in (4) can uniquely measure the maximal convergence rate of quadratic stability of matrix $A_i$ in some sense. Actually, noticing that $P_0/\lambda_2(M(P)) \in \mathcal{D}$, and by (4) we have
\[
\gamma = \lambda_2 \left( A_i^T P_0/\lambda_2(M(P_0)) + P_0/\lambda_2(M(P_0)) A_i \right) \leq -\sigma_i(S).
\]
This shows that, for any PD matrix $P_0$, if $A_i$ is quadratic stable, then the convergent rate is not greater than $-\sigma_i(S)$.

**Definition 4.** For System (2), given matrix set $\mathcal{A} = \{A_i, i \in I\}$ and switching law $\sigma(t)$, let $\mathcal{D}_\sigma^+(s, t; \mathcal{A})$ and $\mathcal{D}^+(s, t; \mathcal{A})$ denote, respectively, the total time running on all stable subsystems and the total time running on all unstable subsystems of System (2) in $[s, t]$; and for $\forall \sigma^+ \in (0, -\sigma^+]$, $\tau^\sigma > 0$, define
\[
\mathcal{D}_1[\sigma^+, \tau^\sigma; \mathcal{A}] = \left\{ \sigma(\cdot): \sup_{t > t_0} \int_{t_0}^t \frac{T_{\sigma^+}(s, t; \mathcal{A})}{T_{\sigma^+}(s, t; \mathcal{A})} \leq \frac{\sigma^+ - \sigma^-}{\sigma^+ - \sigma^-} + \tau^\sigma, \tau_\sigma \geq \tau^\sigma \right\},
\]
\[
\mathcal{D}_2[\sigma^+, \tau^\sigma; \mathcal{A}] = \{ \sigma(\cdot) : \sigma(\cdot) \in \mathcal{D}_1[\sigma^+, \tau^\sigma; \mathcal{A}], \tau_\sigma > \tau^\sigma \}.
\]

**Remark 4.** By Definition 4, for $\forall \sigma(t)$ in $\mathcal{D}_1[\sigma^+, \tau^\sigma; \mathcal{A}]$ (or $\mathcal{D}_2[\sigma^+, \tau^\sigma; \mathcal{A}]$), the ratio of $T_{\sigma^+}^+(s, t; \mathcal{A})$ to $T_{\sigma^+}^+(s, t; \mathcal{A})$ has an upper bound $(\sigma^+ - \sigma^-)/(\sigma^+ + \sigma^-)$, or equivalently,
\[
a_+ T_{\sigma^+}^+(s, t; \mathcal{A}) - a_- T_{\sigma^-}^+(s, t; \mathcal{A}) \\
\leq -\sigma^+ (t - s) \quad \forall t > t_0.
\]
In particular, when all $A_i (i \in I)$ are stable, $\mathcal{D}_1[\sigma^+, \tau^\sigma; \mathcal{A}]$ (and $\mathcal{D}_2[\sigma^+, \tau^\sigma; \mathcal{A}]$) includes all $\sigma(t)$ with average dwell-time no less than (correspondingly, larger than) $\tau^\sigma$.

**Lemma 1.** Consider the following stochastic system:
\[
dx(t) = Ax(t)dt + \sigma(t, x(t))dW(t), \quad x(t_0) = x_0,
\]
where $A \in \mathbb{R}^{n \times n}$ is a constant matrix, $W(t)$ and $\sigma(t, x)$ are similar to (1). If the initial value $x_0$ satisfies $E\|x_0\|^2 < \infty$ and $\exists \sigma(t, x)$ $(r = 1, 2, \ldots, k)$ satisfies
\[
\|\sigma(t, x(t))\| \leq c\|x(t)\| + b(t) \quad \forall t \geq t_0,
\]
(9)
where $c$ is a nonnegative constant, $b(t)$ is random process making $E b^2(t)$ locally Lebesgue integrable on $[t_0, \infty)$, then the $[x(t), t \geq t_0]$ of this system is existent and unique, and for any PD matrix $P \in \mathbb{R}^{n \times n}$,
\[
E[x_t(t)P(t)x(t)] \leq e^{\alpha(t-t_0)}E[x_0(t_0)P(t_0)] + 2k\lambda_M(P)E\int_{t_0}^t e^{\alpha(t-s)}E b^2(s)ds,
\]
(10)
where $a = \lambda_2(M(P) + A^T P) + 2k\lambda_M(P)c^2/\lambda_m(P)$.

**Proof.** From Friedman (1975) it can be seen that $\{x(t), t \geq t_0\}$ is existent and unique. Then, notice the independent increment property of Brownian motion, by Itô formula and Bellman–Gronwall inequality, one can get (10).

**Corollary 1.** Under the conditions of Lemma 1 with $b(t) \equiv 0$ in (9), then $E[x_t(t)P(t)x(t)] \leq E[x_0(t_0)P(t_0)]e^{\alpha(t-t_0)}$, where $a = [\lambda(M(P) + A^T P) + k\lambda_2(P)c^2]/\lambda_m(P)$.

**Lemma 2.** For System (2), if $E\|x(t)\|^2 < \infty$ and $\sigma(t, x)$ satisfies (9), then for $\forall \sigma(t)$ in $\mathcal{D}_1[\sigma^+, \tau^\sigma; \mathcal{A}]$, the $[x(t), t \geq t_0]$, of System (2) is existent, unique and has
\[
E\|x(t)\|^2 \leq e^{\alpha(t-t_0)/2}E\|x_0\|^2 + 2k \int_{t_0}^t e^{\alpha(t-s)}E b^2(s)ds,
\]
(11)
where $\Pi(s, t) = hN_0(s, t) + (2k\alpha^2 - \sigma(t-t))h_\sigma, h_\sigma, h_\sigma, h$ and $c$ are given by (6)–(9).

**Proof.** The existence and uniqueness of solution process $\{x(t), t \geq t_0\}$ follows from Friedman (1975), so we need only to prove (11). For any given $t > t_0$, suppose that $\sigma(t)$ has $j$ switching points in $[t_0, t)$, and denote these $j$ switching points by $t_1, \ldots, t_j$, respectively; and for $l = 0, 1, \ldots, j$, let $\rho_l = [\lambda_2(M(P_{\sigma(t)}) + P_{\sigma(t)}A_{\sigma(t)})] + 2k\lambda_2^2/\lambda_m(P_{\sigma(t)})$. Noticing that $\lambda_2(P_{\sigma(t)}) = 1$, by Lemma 1 we have $E[x_t(t)P_{\sigma(t)}(t)x(t)] \leq e^{\lambda_2(t-t_l)}E\|x(t_j)\|^2 + 2k \int_{t_l}^t e^{\lambda_2(t-s)}E b^2(s)ds$, which together with (7) gives
\[
E\|x(t)\|^2 \leq \lambda_m^{-1} \left[ e^{\lambda_2(t-t_l)}E\|x(t_j)\|^2 + 2k \int_{t_l}^t e^{\lambda_2(t-s)}E b^2(s)ds \right],
\]
(12)
Similarly, for \( l = 1, 2, \ldots, j \) we can obtain
\[
E\|x(t_i)\|^2 \leq \lambda_i^{-1} \left( e^{\mu_i(t_i-(t_{i-1})-1)} E\|x(t_{i-1})\|^2 + 2k \int_{t_{i-1}}^{t_i} e^{\mu_i(t-s)} E(x(s))^2 ds \right).
\]

(13)

Hence, it follows from (12) to (13) that
\[
E\|x(t_i)\|^2 \leq \lambda_i^{-1} e^{\mu_i(t_i-(t_{i-1})-1)} E\|x(t_{i-1})\|^2 + 2k \lambda_i^{-1} \int_{t_{i-1}}^{t_i} e^{\mu_i(t-s)} E(x(s))^2 ds + 2k \sum_{l=0}^{j-1} \lambda_i^{j-l-1} \int_{t_l}^{t_{l+1}} e^{\mu_i(t_{l+1}-s)} E(x(s))^2 ds ds + 2ke^{\mu_i t_j} \int_{t_j}^{\infty} e^{hN(s,t)} E(x(s))^2 ds ds.
\]

Thus, by Remark 4 we have (11). For any given \( hN \), (11) is satisfied with
\[
\|x(t_i)\|^2 \leq \frac{1}{\lambda_i} e^{\mu_i(t_i-(t_{i-1})-1)} E\|x(t_{i-1})\|^2 + 2k \frac{1}{\lambda_i} \int_{t_{i-1}}^{t_i} e^{\mu_i(t-s)} E(x(s))^2 ds + 2ke^{\mu_i t_j} \int_{t_j}^{\infty} e^{hN(s,t)} E(x(s))^2 ds ds.
\]

From Definition 4, \( \sigma(t) \in \mathcal{S}_2[a^*, h\lambda_m/(a^* - 2k^2); c^*] \) implies that \( \tau_{\sigma} > h\lambda_m/(a^* - 2k^2) \), or equivalently, \( h/\tau_{\sigma} > (a^* - 2k^2)/\lambda_m < 0 \). This together with (11) and \( C = \sup_{t_i \geq 0} E(x(t_i))^2 < \infty \) gives \( E\|x(t_i)\|^2 \leq e^{hN(t_i)} E\|x(0)\|^2 + 2k \int_{t_{i-1}}^{t_i} e^{hN(s,t)} E(x(s))^2 ds ds < \infty \). Thus, System (2) is MS stable.

(ii) Notice that for any \( \sigma(t) \in \mathcal{S}_1[a^*, h\lambda_m/(a^* - 2k^2); c^*] \) and \( t - s \geq 0 \), we have (16) and \( h/\tau_{\sigma} > (a^* - 2k^2)/\lambda_m \). Thus, System (2) is MS stable. □

Remark 5. By Definition 4, \( \sigma(t) \in \mathcal{S}_2[a^*, h\lambda_m/(a^* - 2k^2); c^*] \) in Theorem 1 (i) implies that the average dwell-time \( \tau_{\sigma} \) of \( \sigma(t) \) satisfies the inequality \( h\lambda_m/(a^* - 2k^2) \), which gives a lower bound of \( \tau_{\sigma} \). This lower bound is weakened to \( h\lambda_m/(a^* - 2k^2) \) in Theorem 1 (ii). Such a condition seems unavoidable even for the switched deterministic system case.

Example 1. Consider system \( \dot{x}(t) = A_N x(t) \) with \( x(0) = x_0 \). Here, \( A_N \) are given by (6)–(8). By some simple calculations, one can see that
\[
\lim_{t \to \infty} \|x(t)\| = \infty,
\]

if
\[
\sigma(t) = \begin{cases} 1, & t \in [2l, 2l+1), \smallskip \{2, & t \in [2l+1, 2l+2); \smallskip \end{cases}
\]

and
\[
\lim_{t \to \infty} \|x(t)\| = 0,
\]

if
\[
\sigma(t) = \begin{cases} 1, & t \in [6l, 6l+3), \smallskip \{2, & t \in [6l+3, 6l+6), \smallskip \end{cases}
\]

where \( l = 1, 2, \ldots \).

This shows that for general switched systems, the whole systems may be unstable even if its all subsystems are stable, unless some limitations to lower bound of \( \tau_{\sigma} \) of the switching law \( \sigma(t) \) are imposed.

Theorem 2. In addition to the conditions of Theorem 1,

(i) if the condition (15) on \( b(t) \) is replaced by
\[
\int_{t_0}^{\infty} \exp \left( \frac{a^*-2k^2}{\lambda_m} \cdot \frac{t}{\lambda_m} \right) dt < \infty,
\]

then for any \( a^* \in (2k^2, a^-] \) and \( \sigma(t) \in \mathcal{S}_1[a^*, \tau^*; \mathcal{A}^*] \), System (2) is EMS stable with convergent rate \( \tau^* \). Here, \( a \in (0, (a^* - 2k^2)/\lambda_m - a) \), and \( a^-, \lambda_m, a \) are given by (6)–(8).

(ii) if the condition on \( \sigma(t, x(t)) \) is strengthened to
\[
\|x_{\sigma}(t, x(t))\| \leq c \|x(t)\|,
\]

where \( c \) is a nonnegative constant satisfying \( k^2 < a^-\), then for any \( a^* \in (2k^2, a^-] \) and \( \sigma(t) \in \mathcal{S}_1[a^*, \tau^*; \mathcal{A}^*] \), System (2) is EMS stable with convergent rate \( \tau^* \). Here, \( a \in (0, (a^* - 2k^2)/\lambda_m - a) \), and \( a^-, \lambda_m, a \) are given by (6)–(8).
Proof. The existence and uniqueness of \( \{x(t), \forall t \geq t_0\} \) follows from Friedman (1975), so it suffices to show the EMS stability of System (2).

(i) Take arbitrarily \( P_i \in \mathcal{P}_i \ (i \in \mathcal{I}) \). By Definition 4 and \( \tau^* = h((a^* - 2k^2)\lambda_m - a) \), for any \( \sigma(t) \in \mathcal{J}_1[a^*, \tau^*; \mathcal{A}] \), we have \( \tau_{\sigma} \geq \tau^* \), or equivalently, \( a \leq (a^* - 2k^2)\lambda_m - h/\tau_{\sigma} \). This together with (3) gives \( II(x, s) \leq h(0 - a(t - s)) \). Then, from (11) and \( 0 < a < (a^* - 2k^2)/\lambda_m < (a^* - 2k^2)/\lambda_m \) it follows that \( E \left\| x(t) \right\|^2 e^{-(1+t_0)h(0-a(t-t_0))} \leq E \left\| x_0 \right\|^2 + 2k \int_{t_0}^{t} e^{h(t-s)} E^2(s) ds \leq E \left\| x_0 \right\|^2 + 2k \int_{t_0}^{t} e^{(a^*-2k^2)\lambda_m(t-t_0)} E^2(s) ds \). Hence, by (18) and \( a > 0 \) we know that System (2) is EMS stable with convergent rate \( a \).

(ii) For any given \( t \geq t_0 \), suppose that \( \sigma(t) \) has \( j \) switching points in the time interval \( [0, t) \), and denote these \( j \) switching points by \( t_1, t_2, \ldots, \text{and} \ t_j \), respectively. For, \( \forall j = 0, 1, \ldots, j \), let \( \mu_j = [\lambda_M(P_{\sigma(t)}) + P(\sigma(t)) + A_{\sigma(t)}] + k^2 / \lambda_m \). Then, from (7) \( \lambda_M(P_{\sigma(t)}) = 1 \), by Corollary 1 we have \( E \left\| x(t) \right\|^2 \leq \lambda_m \mu_j \| x(t_j) \|^2 \). Further, by the definition of \( \lambda_m \) we have \( E \left\| x(t_j) \right\|^2 \leq \lambda_m^{-1} \mu_j \| x(t_j-1) \|^2 \). Similarly, we can get

\[
E \left\| x(t_j) \right\|^2 \leq \lambda_m^{-1} e^{\mu_j(t_j-t_{j-1})} E \left\| x(t_{j-1}) \right\|^2, \\
E \left\| x(t_0) \right\|^2 \leq \lambda_m^{-1} e^{\mu_0(t_0-t_{-1})} E \left\| x(t_{-1}) \right\|^2.
\]

Hence, \( E \left\| x(t) \right\|^2 \leq \lambda_m^{-1} e^{\mu_0(t_0-t_1)+\mu_1(t_1-t_2)+\ldots+\mu_j(t_j-t_{j-1})} E \left\| x(t_{j-1}) \right\|^2 \). By (8) and \( \lambda_M = 1 \), we have \( \lambda_m^{-1} = e^0 \), which together with \( N_{\sigma}(t_0, t) = j \) gives \( \lambda_m^{-1} e^{hN_{\sigma}(t_0, t)} \).

From (4) to (7) it follows that \( \hat{\lambda}_M(A^T_{\sigma(t)}P_{\sigma(t)} + P_{\sigma(t)}A_{\sigma(t)}) \leq a^+ \) when \( A_{\sigma(t)} \subseteq C^- \), and \( \hat{\lambda}_M(A^T_{\sigma(t)}P_{\sigma(t)} + P_{\sigma(t)}A_{\sigma(t)}) \leq a^+ \) when \( A_{\sigma(t)} \subseteq C^- \). Then, by Definition 4 we have \( \mu_0(t_0-t_0) + \mu_1(t_1-t_1) + \ldots + \mu_{j-1}(t_{j-1}-t_j) \leq a^+T^+(t_0, t_0; \mathcal{A}) \leq a^+T^+(t_0, 0; \mathcal{A}) + k^2 (t-0)/\lambda_m \). This implies \( E \left\| x(t) \right\|^2 \leq e^{h(hN_{\sigma}(t_0, t) + k^2 (t_0-t_0)/\lambda_m + [a^+T^+(t_0, 0; \mathcal{A}) + k^2]}/\lambda_m E \left\| x_0 \right\|^2 \). Together with Remark 4 leads to

\[
E \left\| x(t) \right\|^2 \leq e^{h(hN_{\sigma}(t_0, t) + k^2 (t_0-t_0)/\lambda_m)} E \left\| x_0 \right\|^2. \tag{20}
\]

By Definition 4 and

\[
\tau^* = h((a^* - 2k^2)/\lambda_m - a)
\]

we see that for \( \forall \sigma(t) \in \mathcal{J}_1[a^*, \tau^*; \mathcal{A}] \), \( \tau_{\sigma} \geq \tau^* \), or equivalently, \( a \leq (a^* - 2k^2)/\lambda_m - h/\tau_{\sigma} \). This together with (3) and (20) gives

\[
E \left\| x(t) \right\|^2 \leq e^{h(1+t_0)-a(t-t_0)} E \left\| x_0 \right\|^2.
\]

Thus, System (2) is EMS stable with convergent rate \( a \).

Remark 6. In fact, for any PD matrix \( P_i (i \in \mathcal{I}) \), if all the conditions of Theorems 1 and 2 hold with \( a^+, a^-, \lambda_m \) and \( k^2 \) replaced by

\[
\max_{1 \leq i \leq N} \max_{1 \leq i \leq N} (\hat{\lambda}_M(P_iA_i + A_i^TP_i) + 0), \\
\min_{1 \leq i \leq N} \min_{1 \leq i \leq N} (\hat{\lambda}_M(P_iA_i + A_i^TP_i), 0),
\]

we have \( \lambda_m(P_i) \) and \( k^2 \max_{1 \leq i \leq N} \hat{\lambda}_M(P_i) \), respectively, then the conclusions of Theorems 1–2 are still true.

All the above results have their corresponding corollaries. Here, we give only two corollaries to Theorem 2 (i).

Corollary 2. For System (2), if \( A_i + A_i^T < 0 \ (i \in \mathcal{I}) \), and (18) holds with \( a^* \) replaced by \( \mu \triangleq \min_{1 \leq i \leq N} |\hat{\lambda}_M(A_i + A_i^T)| \), then under the conditions of Theorem 2 (i), System (2) is EMS stable with convergent rate \( \mu - 2k^2 \) for all \( \sigma(t) \).

Proof. Noticing that the definition of \( \mu \), and for this corollary, since \( A_i + A_i^T < 0 \) we have \( T^+(s, t; \mathcal{A}) \equiv 0 \), \( a^+ = 0 \) and \( a^* = \mu \), and can take \( P_i \equiv I (i \in \mathcal{I}) \) which implies \( \lambda_m = \lambda_M = 1 \). Similar to the proof of Lemma 2, we know System (2) is EMS stable with convergent rate \( \mu - 2k^2 \).

Corollary 3. For System (2), in addition to the conditions of Theorem 2 (i), suppose \( A_1 \subset C^- (i \in \mathcal{I}) \), and \( A^T_{i-1} + A_i < 0 \) for \( p = 1, \ldots, N_1 \). Let

\[
\mu = \min_{1 \leq p \leq N_1} \left( \min_{1 \leq i \leq N} |\hat{\lambda}_M(A_i + A_i^T)|, \min_{N_1+1 \leq p \leq N} |\lambda_{i}| \right)
\]

with \( a_i \) given in (4). Then the conclusion of Theorem 2 (i) is true for any \( a \in (0, (\mu - 2k^2)/\min(\lambda_m, 1)) \) and \( \sigma(t) \), provided that (18) holds with \( a^* \) replaced by \( \mu \), and the average dwell-time \( \tau_{\sigma} \) on \( \{A_1, A_2, \ldots, A_{N_1}\} \) satisfies \( \tau_{\sigma} \geq h/(\mu - 2k^2)/\min(\lambda_m, 1) \) – 1).

Remark 7. Corollary 2 shows that when \( A_1 (i \in \mathcal{I}) \) have a common Lyapunov matrix \( P = I \), there is no need to make any restriction to \( \sigma(t) \) to ensure the EMS stability of the system, namely the switch can be arbitrary. This remark is also valid when \( P \neq I \).

Remark 8. Corollary 3 shows that for any \( \sigma(t) \) there is no need to impose any restriction to the switching frequency of \( \sigma(t) \) on \( \{A_1, \ldots, A_{N_1}\} \).

4. Stabilization control of SS systems

Based on Theorems 1–2, we can establish a series of stabilization results. Here, for simplicity, we will discuss only the ones corresponding to Theorem 2 (ii), that is, under the assumption (19), we would like to design a state feedback control of the form \( u_{\sigma(t)}(t) = K_{\sigma(t)}x(t) \) such that the closed-loop system

\[
dx(t) = A_{\sigma(t)}x(t) dt + \sigma(t, x(t)) dW(t), \quad x(t_0) = x_0 \tag{21}
\]

is EMS stable, where \( \tilde{A}_{\sigma(t)} = A_{\sigma(t)} + B_{\sigma(t)}K_{\sigma(t)} \).
In the following, we denote all $K_i$ by $K$ with proper dimension, for short.

Let $\mathcal{A} = \{A_i, i \in \mathcal{I}\}$. Obviously, if all $A_i$ in $\mathcal{A}$ are stable, then for any given PD matrix $Q_i$, there exists a PD matrix $P_i(K)$ such that
\[
\tilde{A}_i^T P_i(K) + P_i(K) \tilde{A}_i = -Q_i \quad \forall i \in \mathcal{I}.
\] (22)

By Lemma 3.2 of Cheng et al. (2005), for a controllable subsystem $(A_i, B_i)$ of System (1), one can arbitrarily change the $\hat{\lambda}_M(P_i(K))$ with proper feedback matrix $K$; but for an UCB$S$ subsystem $(A_i, B_i)$ of System (1), the case is different. This means that one has to explicitly characterize the dominant impact of the uncontrollable eigenvalue of $(A_i, B_i)$ on $\hat{\lambda}_M(P_i(K))$. To do so, set
\[
\mathcal{H}(A_i, B_i) = \{K : \nu(\tilde{A}_i, B_i) \leq \mu(A_i, B_i)\},
\] (23)
where $\nu(\tilde{A}_i, B_i)$ is the largest real part of all controllable eigenvalues of $(A_i, B_i)$, and $\mu(A_i, B_i)$ denotes the largest negative real part of all uncontrollable eigenvalues of $(A_i, B_i)$.

Then, for an UCB$S$ subsystem $(A_i, B_i)$, we define
\[
\eta^{(i)}_M = \min_{K \in \mathcal{H}(A_i, B_i)} \hat{\lambda}_M(P_i(K)),
\]
\[
\eta^{(i)}_m = \lambda_\text{min}(P_i(K)).
\] (24)

Further, for System (1), suppose that subsystems $(A_{ip}, B_{ip})$ are controllable for $p = 1, \ldots, N_1 (\leq N)$, and are UCB$S$ for $p = N_1 + 1, \ldots, N$. For the PD matrices $Q_i$ and $P_i(K)$ in (22), let $\Psi_i = \hat{\lambda}_M(\tilde{A}_i^T P_i(K) + P_i(K) \tilde{A}_i)$; and similar to (4)–(6), introduce
\[
\begin{align*}
a^+(\mathcal{A}) &= \max_{i \in \mathcal{I}} \min_{i \in \mathcal{I}} |\psi_i^*(0)|, \\
a^-(\mathcal{A}) &= \min_{i \in \mathcal{I}} \max_{i \in \mathcal{I}} |\psi_i^*(0)|, \\
\eta_{M(i)} &= \max_{1 \leq p \leq N_1} \hat{\lambda}_M(P_{ip}(K)), \\
\eta_{m(i)} &= \min_{1 \leq p \leq N_1} \lambda_\text{min}(P_{ip}(K)), \\
\end{align*}
\]
where $\psi_i^*(s)$, $\lambda_\text{min}(s)$ are given in (26)–(28).

Replacing $a^-$ and $kc^2$ with $a^-(\mathcal{A})$ and $kc^2 \eta_M$, respectively, in Theorem 2(ii), by Remark 5 we can get

**Lemma 3.** If $\sigma(t, x)$ satisfies (19), and there are $K_i (i \in \mathcal{I})$ such that all the $\tilde{A}_i = A_i + B_i K_i$ are stable and the PD matrices $P_i(K_i)$ and $Q_i (i \in \mathcal{I})$ satisfy
\[
kc^2 \eta_M < a^-(\mathcal{A}),
\] (27)
then for any $\sigma(t) \in \mathcal{I}_i[a^-(\mathcal{A}), \tau_a; \mathcal{A}]$, System (21) is EMS stable with convergent rate $a$. Here, $a \in (0, (a^-(\mathcal{A}) - kc^2 \eta_M)/\eta_m)$, $\tau_a = (\ln(\eta_M/\eta_m))/((a^-(\mathcal{A}) - kc^2 \eta_M)/\eta_m - a)$, and $a^-(\mathcal{A}), \eta_M$ and $\eta_m$ are given in (25)–(26).

**Remark 12.** Stabilizing System (21) is not essentially dependent on the choice of PD matrices $Q_i$, although the definitions of $a^-(\mathcal{A}), \eta_M$ and $\eta_m$ in Lemma 3 and (25)–(26) are obviously related to the choice of $Q_i$. In particular, when $\eta_M = \eta_m$ (denote them with $\eta$, for short), Lemma 3 becomes a special case of Corollary 1 (with some changes on the conditions and conclusions in accordance with Theorem 2(ii)): for any given switching law $\sigma(t)$, System (21) is EMS stable with convergent rate $a^-(\mathcal{A})/\eta - kc^2$.

**Definition 5.** For $\forall t \geq 0$, if the value of $\sigma(s)$ is well-defined and available to the control design for all $s \in [t_0, t]$, then the process $\sigma(s)$ is said observable.

**Remark 13.** By Definition 5, if a switching law $\sigma(s)$ is observable, then, at any time instant $t > t_0$, the value $\sigma(s), \forall s \in [t_0, t]$, is given and available to control design, although the information on $\sigma(s) : s > t$ may be unknown. This is the essential difference between the switched system control and the conventional time-varying system control (Anderson $\&$ Moore, 1971). The latter requires that the value of the time process $\sigma(t)$ (and hence, $A_{\sigma(t)}(s)$ and $B_{\sigma(t)}(s)$) on the whole control time interval $[t_0, \infty)$ is completely known at time $t_0$.

**Theorem 3.** For System (1), suppose that subsystems $(A_i, B_i)$ $(i \in \mathcal{I})$ are controllable (that is, $N_1 = N$ in (26)), $\sigma(t, x)$ satisfies (19), switching law $\sigma(s)$ is observable and $\sigma(s) \in \mathcal{S}[\tau^*, N_0]$ for some given constants $N_0 \geq 0$ and $\tau^* > 0$, then there exists a state feedback $u_{\sigma(t)}(t) = K_{\sigma(t)} x(t)$ such that System (21) is EMS stable with convergent rate $a \in (0, [1 - kc^2 \eta_M]/\eta_m)$. Here, $\tau^*, N_0$ and $\tau^*$ are given in Definition 3, $\eta_m$ and $\eta_M$ in (26).

**Proof.** By Remark 12, one can simply take $Q_i = I$ $(i \in \mathcal{I})$. In this case, we have $a^-(\mathcal{A}) = 1$.

Since $(A_i, B_i)$ are controllable, there exist feedback matrices $K_i \in \mathbb{P}^{m_i \times ni}$ such that $A_i(\tilde{A}_i) \subset (-\infty, 0)$, where $\tilde{A}_i = A_i + B_i K_i$. By Lemma 3.2 of Cheng et al. (2005) we can obtain
\[
\hat{\lambda}_M(P_i) = \|P_i\| = \|\int_0^\infty e^{A_i t} e^{A_i t} dt\| \leq (M^2/2)\lambda^2 + n - 3.
\] Thus, without loss of generality, we assume that $K_i$ has already been chosen such that $\eta_m$ and $\eta_M$ in (26) are sufficiently small so that
\[
kc^2 \eta_M + \frac{\eta_m}{\tau^2} \ln \frac{\eta_M}{\eta_m} < 1 = a^-. 
\] (28)
In fact, noticing \(\lim_{\eta \to 0^+} \eta \ln \eta = 0\), from the straightforward inequality \((\eta_M/\eta^*) \ln(\eta_M/\eta_m) \leq (1/\eta^*)[\ln(\eta_M/\eta_m) + \ln(\eta_M/\eta_m)]\) we know that when \(\eta_m\) and \(\eta_M\) are sufficiently small, (28) must be true.

In this case, if \(\eta_m = \eta_M = \eta\), then by Remark 12, System (21) is EMS stable with convergent rate \(1/\eta - kc^2\); when \(\eta_m \neq \eta_M\), let \(a = (1 - kc^2\eta_M)/\eta_m - (1/\eta^*)\ln(\eta_M/\eta_m)\). Then, by (28) one can see that \(a \in (0, 1 - kc^2\eta_M)/\eta_m\) and the average dwell-time \(\tau^*\) of \(\mathscr{D}[\tau^*, N_0]\) can be expressed as \(\tau^* = (\ln(\eta_M/\eta_m))/((1 - kc^2\eta_M)/\eta_m - a)\). Hence, by Definitions 3–4, we have \(\sigma(t) \in \mathscr{D}_1[1, \tau^*, \mathscr{A}]\). Thus, by Lemma 3, System (21) is EMS stable with convergent rate \(a\). \(\square\)

**Remark 14.** By Theorem 3, for any given constant \(c > 0\) in (19), System (21) can be stabilized by state feedback control if all \((A_i, B_i)\) \(i \in \mathscr{J}\) are controllable. However, when some subsystem is uncontrollable, in order to stabilize System (1), some restriction to the constant \(c\) in (19) seems necessary. To see this, present the following example.

**Example 2.** Assume \((A, B)\) is a subsystem of a switched system with

\[
A = \begin{pmatrix} -2 & 0 \\ 0 & -1 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 1 \\ 0 \end{pmatrix}.
\]

Obviously, \((A, B)\) is uncontrollable. For any given feedback matrix \(K = (k_1, k_2)\), the state matrix of the closed-loop system is with the form

\[
\tilde{A} = A + BK = \begin{pmatrix} k_1 - 2 & k_2 \\ 0 & -1 \end{pmatrix},
\]

which gives

\[
e^{\tilde{A}t} = \begin{pmatrix} e^{(k_1 - 2)t} & \ast \\ 0 & e^{-t} \end{pmatrix},
\]

where \(\ast\) denotes a function of \(t\) depending on \(k_1\) and \(k_2\). When \(K\) is such that \(A(K) \subset (-\infty, 0)\) (in this case, \(k_1 < 2\) and \(k_2\) can be an arbitrary real number), the solution matrix \(P\) of the Lyapunov equation \(\tilde{A}^TP + PA = -I\) can be expressed as

\[
P = \int_0^\infty e^{\tilde{A}t} e^{\tilde{A}t^*} \, dt = \begin{pmatrix} 1/(2(k_1 - 2)) & \ast \\ \ast & 1/2 + \int_0^\infty s^2 \, ds \end{pmatrix},
\]

where \(\ast\) is a nonnegative constant depending on \(k_1\) and \(k_2\). Hence, we have \(\lambda_M(P) = \|P\| \geq 1/2 + \int_0^\infty s^2 \, ds \geq 1/2\). This implies that the \(\eta_M\) in (26) is greater than \(1/2\) for all state feedback controls. Thus, in order to get the condition \(1 - kc^2\eta_M > 0\), \(c\) must be less than \(1/\sqrt{k\eta_M}\), where \(k\) is the dimension of the noise coefficient matrix \(v(t, x(t))\).

For the case where only a part of subsystems of the System (1) is controllable, we have the following stabilization theorem.

**Theorem 4.** For System (1), suppose that subsystems \((A_{i_p}, B_{i_p})\) are controllable for \(p = 1, \ldots, N_1(< N)\), and UChS for \(p = N_1 + 1, \ldots, N\), \(z(t, x(t))\) satisfies (19), switching law \(s(t)\) is observable and \(s(t) \in \mathscr{D}[\tau^*, N_0]\) for some constants \(N_0 \geq 0\) and \(\tau^* > 0\). If \(1 - kc^2\max_{1 \leq p \leq N} \eta_{i(p)}^{(i)} > 0\), then there exists a state feedback \(u_{i(p)}(t) = K_{i(p)}x(t)\) such that System (21) is EMS stable with convergent rate \(a \in (0, 1 - kc^2\eta_M)/\eta_m\).

Here, \(\mathscr{D}[\tau^*, \mathscr{A}]\) and \(\eta_{i(p)}^{(i)}\) is given in Definition 3 and (24), \(\eta_m\) and \(\eta_M\) in (26).

**Proof.** Similar to Theorem 3, for (22), choose \(Q_i = I\) \((i \in \mathscr{J})\). Then, \(u = \rho\) and \(B\). Thus, by Remark 12, System (21) is EMS stable with convergent rate \(a\). \(\square\)

5. Concluding remark

By introducing two compact matrix sets \(\mathscr{D}\) and \(\mathscr{D}_2\), we have studied the MS stability of a class of switched system with stochastic disturbances. Based on the concept of average dwell-time and the ratio of the total time running on all unstable subsystems to the total time running on all stable subsystems, some sufficient conditions on MS stability and EMS stability are given, respectively. The impact of the uncontrollable part of an UChS stochastic system on the convergent rate of the closed-loop system is analyzed, and the stabilization control problems of two classes of SS control systems are investigated. One is with all subsystems controllable, and the other is with some subsystems controllable and some UChS. Some sufficient conditions on the EMS stability of the closed-loop systems, and the convergent rates of the closed-loop systems are given. Unlike conventional time-varying parameter systems, here it is only required that the switching law is observable.
References


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